THE EXISTENCE OF IRRATIONAL NUMBERS

The existence of irrational numbers is a classical result that has already been known to the old Greeks. The proof works by contradiction and is a good example of mathematical reasoning.

Theorem 0.1.
There is no rational number $a \in \mathbb{Q}$ whose square equals 2.

Proof. Let us assume that there exists $a \in \mathbb{Q}$ such that $a^2 = 2$. We show that this assumption leads to a contradiction, from which we conclude that no such $a \in \mathbb{Q}$ can exist.

Since $a \in \mathbb{Q}$ by assumption, there exist $p, q \in \mathbb{Z}$ with $q \neq 0$ such that $a = p/q$. We may assume that p and q are both non-negative. We may also assume that $\frac{p}{q}$ is a proper fraction, i.e., p and q do not share a common integer divisor other than 1.

We now see that

$$2 = a^2 = \frac{p^2}{q^2}$$

and thus

$$2q^2 = p^2.$$

Hence p^2 is an even number. But when the square of a number is an even number, then that number must have been even in the first place. Hence we conclude that p is an even number too.

Next, since p is an even number, we may write it as $p = 2r$ for some non-negative integer $r \in \mathbb{N}_0$. Consequently,

$$2q^2 = p^2 = 4r^2$$

which implies that

$$q^2 = 2r^2.$$

In particular, q^2 is an even number. Arguing analogously as have done for p above, we get that q is an even number in the first place.

We have derived that p and q are even numbers, so both of them are divisible by 2. But we have assumed above that p and q do not share a common divisor. Hence, from our initial assumption that there exists $a \in \mathbb{Q}$ with $a^2 = 2$ we have derived that two mutually exclusive statements are true at the same time, which is a contradiction.

It thus follows that there cannot exist $a \in \mathbb{Q}$ with $a^2 = 2$. □

Remark 0.2.
One might argue that the proof above still needs more explanation: we have used that if the square of a number is even, then the original number must have been even too in the first place. One may argue by the uniqueness of the prime factorization, but another more elementary approach works as follows.

Let $a \in \mathbb{N}_0$ such that a^2 is even. If the last decimal digit of a is 1, 3, 5, 7, or 9, then the last digit of a^2 cannot be 0, 2, 4, 6, or 8. Hence, if a is an odd number, then a^2 is not even. So, if a^2 is even, then a is not odd, which means that a is even.

Note that this little mathematical reasoning again used a proof by contradiction.