Please check the website for your room assignment!

General remarks:

- You will be asked to write down your name and student ID on the front page.
- The Final takes place 3:00-5:59 pm.
- No books, calculators, phones, or cheatsheets are allowed during class.
- Homework 1: Exercise 2,4,5
- Homework 2: Exercise 1,2
- Homework 3: Exercise 1,2,3,4
- Homework 4: Exercise 1
- Homework 5: Exercise 1, 2
- Homework 6: Exercise 1, 2
- Homework 7: Exercise 1, 2, 3, 4
- Homework 8: Exercise 1, 2, 3, 4
- Homework 9: Exercise 1, 2, 3, 4
- **Repeat the content of Midterms 1 and 2, and the matrix workout.**

Exercise 1.
Consider a linear mapping $L : \mathbb{R}^3 \to \mathbb{R}^4$ such that

$$\begin{aligned} L(e_1) &= 5e_1 + 3e_2 + 1e_3 - 4e_4 \\ L(e_2) &= 6e_1 + 0e_2 + 2e_3 - 1e_4 \\ L(e_3) &= 2e_1 - 2e_2 + 1e_3 - 2e_4 \end{aligned}$$

Find a matrix $A \in \mathbb{R}^{4\times 3}$ such that $L = L_A$.

Solution 1.

$$A = \begin{pmatrix} 5 & 6 & 2 \\ 3 & 0 & -2 \\ 1 & 2 & 1 \\ -4 & -1 & -2 \end{pmatrix}$$

Exercise 2.
Let $\mathcal{P}_3$ and $\mathcal{P}_4$ be the vector spaces of polynomials of maximal degree 3 and 4, respectively. The monomials $1, x, x^2, x^3$ are a basis of $\mathcal{P}_3$ and the monomials $1, x, x^2, x^3, x^4$ are a basis of $\mathcal{P}_4$.

(1) The inclusion $I : \mathcal{P}_3 \to \mathcal{P}_4$ is a linear mapping. Which matrix represents it?

(2) Consider the linear mapping

$$\partial : \mathcal{P}_4 \to \mathcal{P}_3$$

that satisfies

$$\partial(1) = 0, \quad \partial(x) = 1, \quad \partial(x^2) = 2x, \quad \partial(x^3) = 3x^2, \quad \partial(x^4) = 4x^3.$$

What is the value of

$$\partial\left(a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0\right)?$$

(3) Represent the linear mapping $\partial : \mathcal{P}_4 \to \mathcal{P}_3$ by a matrix with respect to the monomial bases.

(4) Represent the linear mapping $\partial : \mathcal{P}_4 \to \mathcal{P}_3$ by a matrix with respect to the monomial basis $1, x, x^2, x^3, x^4$ of $\mathcal{P}_4$ and the basis $1, 2x, 3x^2, 4x^3$ of $\mathcal{P}_3$.

(5) Consider the polynomials

$$\mathcal{B} := \left\{ \quad 1 + x, \quad x + x^2, \quad x^2 + 1 \quad \right\}.$$

Show that this is a basis of $\mathcal{P}_2$. Try different techniques to prove your result.

(6) Compute the basis transition matrix $A \in \mathbb{R}^{3\times 3}$ such that

$$A \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix}$$

if and only if

$$\alpha_0 \cdot 1 + \alpha_1 x + \alpha_2 x^2 = \beta_0 \cdot (1+x) + \beta_1 (x + x^2) + \beta_2 (1 + x^2).$$

Represent the linear mapping

$$T : \mathcal{P}_2 \to \mathcal{P}_2, \quad \left(a_0 + a_1 x + a_2 x^2\right) \mapsto \left(3a_0 + 4a_1 x + 5a_2 x^2\right)$$

by a matrix with respect to the basis $\mathcal{B}$.

Solution 2. (1)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(2)

$$4a_4 x^3 + 3a_3 x^2 + 2a_2 x + a_1.$$

(3)

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

(4)

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(5) One way to prove that $\mathcal{B}$ is a basis of $\mathcal{P}_2$ is to observe that the members of $\mathcal{B}$ can be represented with respect to the monomial basis as the columns of the matrix

$$B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

A direct computation shows that B has determinant 2, so B is invertible. But that means that the mapping $L : \mathbb{R}^3 \to \mathcal{P}_2$ with

$$L(e_1) = 1 + x, \quad L(e_2) = x + x^2, \quad L(e_3) = 1 + x^2.$$

is invertible. In particular, for every $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ the polynomial

$$L(\alpha_1 e_1 + \alpha_2 e_2 + \alpha e_3) = \alpha_1(1+x) + \alpha_2(x + x^2) + \alpha_3(1 + x^2)$$

is zero if and only if $\alpha_1 = \alpha_2 = \alpha_3 = 0$. We already know that $\mathcal{P}_2$ has dimension 3, for example, because the three polynomials $1, x, x^2$ form a basis of $\mathcal{P}_2$. Hence the members of $\mathcal{B}$ form a basis of $\mathcal{P}_2$.

Another possible to show this is just matrix manipulation in disguise. Suppose that $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ with

$$\alpha_1(1+x) + \alpha_2(x + x^2) + \alpha_3(1 + x^2) = 0.$$

Using that $1, x, x^2$ is a basis of $\mathcal{P}_2$, we conclude that

$$\alpha_1 + \alpha_3 = \alpha_1 + \alpha_2 = \alpha_2 + \alpha_3 = 0.$$

Elimination gives $\alpha_2 = -\alpha_1$ and $\alpha_3 = -\alpha_1$, and so $\alpha_2 = \alpha_3$. It follows that $\alpha_2 + \alpha_2 = 0$. But the latter identity already enforces $\alpha_2 = 0$, and thus $\alpha_3 = \alpha_2 = 0$. Now $\alpha_1 = 0$ is evident.

(6) As already mentioned above, we have

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

if and only if

$$\beta_0 \cdot (1 + x) + \beta_1(x + x^2) + \beta_2(1 + x^2) = \alpha_0 \cdot 1 + \alpha_1 x + \alpha_2 x^2.$$

One computes that

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

Obviously, this is the desired matrix.

Now, the mapping T is represented with respect to the monomial basis by the matrix

$$C = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

To represent it in terms of the basis $\mathcal{B}$ we first transform the coefficients with respect to the basis $\mathcal{B}$ to coefficients with respect to the monomial basis. We apply the matrix C, which expresses the action of T in terms of the monomial basis. Now transforming the coefficients of this result in terms of the monomial basis to the coefficients in terms of the basis $\mathcal{B}$ yields the final result.

In total, this is represented by the matrix product

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 3 & 0 & 3 \\ 4 & 4 & 0 \\ 0 & 5 & 5 \end{pmatrix}$$
$$= \frac{1}{2} \begin{pmatrix} 7 & -1 & -2 \\ 1 & 9 & 2 \\ -1 & 1 & 8 \end{pmatrix}.$$

Exercise 3.
Consider the following matrices:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ 4 & 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -4 & -6 \\ -1 & 1 & -3 \\ 1 & 2 & 6 \end{pmatrix}.$$

(1) Compute the characteristic polynomials of these matrices and determine the eigenvalues.
(2) Use the characteristic polynomial to find the determinants of the matrices.
(3) Use the characteristic polynomial to find the determinants and eigenvalues of the following matrices: $5A$, $3B$, $2C$, $3A + 4\,\mathrm{Id}_3$, $-B + \mathrm{Id}_3$, $C + 7\,\mathrm{Id}_3$.
(4) Find the eigenvectors of the matrices A, B, and C, and write each matrix in the form XDX^{-1} with $X \in \mathbb{R}^{n\times n}$ invertible and $D \in \mathbb{R}^{n\times n}$ a diagonal matrix.

Solution 3. (1)

$$\mu_A(\lambda) = -\lambda^3 + 4\lambda^2 + \lambda - 4 = -(\lambda-4)(\lambda+1)(\lambda-1),$$
$$\mu_B(\lambda) = (\lambda-3)(\lambda-1)(\lambda-5) = \lambda^3 - 9\lambda^2 + 23\lambda - 15,$$
$$\mu_C(\lambda) = -\lambda^3 + 8\lambda^2 - 21\lambda + 18 = -(\lambda-3)^2(\lambda-2).$$

(2) We have

$$\det(A) = \mu_A(0) = 4, \quad \det(B) = \mu_B(0) = -15, \quad \det(C) = \mu_C(0) = -18,$$

(3) We use that scaling a matrix scales its eigenvalues (which should be obvious to you) and that adding a multiple of the identity shifts its eigenvalues (which should be obvious to you, too). Since $p_X(\lambda) = \det(X - \lambda \operatorname{Id}_3)$, the determinants now compute as follows:

$$\det(5A) = 5 \cdot 4 \cdot 5 \cdot 1 \cdot 5 \cdot (-1) = -500,$$
$$\det(3B) = 3 \cdot 3 \cdot 3 \cdot 1 \cdot 3 \cdot 5 = 405,$$
$$\det(2C) = 2 \cdot 3 \cdot 2 \cdot 3 \cdot 2 \cdot 2 = 144,$$
$$\det(3A + 4\operatorname{Id}_3) = (3 \cdot 4 + 4) \cdot (3 \cdot 1 + 4) \cdot (3 \cdot (-1) + 4) = 112,$$
$$\det(-B + \operatorname{Id}_3) = (-3+1)(-1+1)(-1+5) = 0,$$
$$\det(C + 7\operatorname{Id}_3) = (7-3)^2(7-2) = 4^2 \cdot 5 = 80.$$

For any matrix $X \in \mathbb{C}^{n \times n}$ with an eigenvalue $\lambda \in \mathbb{C}$, we have that $\alpha X + \beta \operatorname{Id}_3$ has the eigenvalue $\alpha\lambda + \beta$. To see this, let $v \in \mathbb{C}^n$ be an eigenvector of X; now compute Xv and $(\alpha X + \beta \operatorname{Id}_n)v$.

(4) The eigenvectors can be found by solving homogenous systems of linear equations and are the following:

$$A\begin{pmatrix}2\\5\\6\end{pmatrix} = 4\begin{pmatrix}2\\5\\6\end{pmatrix}, \quad A\begin{pmatrix}-1\\0\\2\end{pmatrix} = -1\begin{pmatrix}-1\\0\\2\end{pmatrix}, \quad A\begin{pmatrix}-1\\2\\0\end{pmatrix} = 1\begin{pmatrix}-1\\2\\0\end{pmatrix},$$
$$B\begin{pmatrix}1\\0\\0\end{pmatrix} = 3\begin{pmatrix}1\\0\\0\end{pmatrix}, \quad B\begin{pmatrix}\frac{2}{3}\\1\\0\end{pmatrix} = 1\begin{pmatrix}\frac{2}{3}\\1\\0\end{pmatrix}, \quad B\begin{pmatrix}1\\-2\\1\end{pmatrix} = 5\begin{pmatrix}1\\-2\\1\end{pmatrix},$$
$$C\begin{pmatrix}-3\\0\\1\end{pmatrix} = 3\begin{pmatrix}-3\\0\\1\end{pmatrix}, \quad C\begin{pmatrix}-2\\1\\0\end{pmatrix} = 3\begin{pmatrix}-2\\1\\0\end{pmatrix}, \quad C\begin{pmatrix}-2\\-1\\1\end{pmatrix} = 2\begin{pmatrix}-2\\-1\\1\end{pmatrix}.$$

The matrices can be diagonalized as follows:

$$A = \begin{pmatrix}2 & -1 & -1\\5 & 0 & 2\\6 & 2 & 0\end{pmatrix}\begin{pmatrix}4 & 0 & 0\\0 & -1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}2 & -1 & -1\\5 & 0 & 2\\6 & 2 & 0\end{pmatrix}^{-1},$$
$$B = \begin{pmatrix}1 & \frac{2}{3} & 1\\0 & 1 & -2\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}3 & 0 & 0\\0 & 1 & 0\\0 & 0 & 5\end{pmatrix}\begin{pmatrix}1 & \frac{2}{3} & 1\\0 & 1 & -2\\0 & 0 & 1\end{pmatrix}^{-1},$$
$$C = \begin{pmatrix}-3 & -2 & -2\\0 & 1 & -1\\1 & 0 & 1\end{pmatrix}\begin{pmatrix}3 & 0 & 0\\0 & 3 & 0\\0 & 0 & 2\end{pmatrix}\begin{pmatrix}-3 & -2 & -2\\0 & 1 & -1\\1 & 0 & 1\end{pmatrix}^{-1}.$$