THE BINOMIAL THEOREM

We prove the Binomial Theorem. This can be thought of as a generalization of the first binomial identity.

Theorem 1
Let $x, y \in \mathbb{R}$ and let $n \in \mathbb{N}$. Then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof. We prove the claim by induction. Let $x, y \in \mathbb{R}$.

First, for the induction base with $n = 1$ we observe

$$(x+y)^1 = x + y = \binom{1}{0} x + \binom{1}{1} y = \sum_{k=0}^{1} \binom{1}{k} x^k y^{1-k}.$$

Second, for the induction step we assume that the claim is true for $n \in \mathbb{N}$ and show that this implies that the claim is true for $n+1$ as well.

We see that

$$(x+y)^{n+1} = (x+y)^n (x+y) = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} (x+y),$$

where we have used the induction assumption. Next we rearrange this as

$$\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} (x+y) = \sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k} + \sum_{k=0}^{n} \binom{n}{k} x^k y^{n+1-k}.$$

We now shift the index in the first sum on the right-hand side above. This gives

$$\sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k} = \sum_{k=1}^{n+1} \binom{n}{k-1} x^k y^{n+1-k}.$$

Plugging this we now have

$$\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} (x+y) = \sum_{k=1}^{n+1} \binom{n}{k-1} x^k y^{n+1-k} + \sum_{k=0}^{n} \binom{n}{k} x^k y^{n+1-k}.$$

The two sums on the right-hand side can now be rearranged as

$$\begin{aligned} &\sum_{k=1}^{n+1} \binom{n}{k-1} x^k y^{n+1-k} + \sum_{k=0}^{n} \binom{n}{k} x^k y^{n+1-k} \\ &= \sum_{k=1}^{n} \binom{n}{k-1} x^k y^{n+1-k} + \binom{n}{n} x^{n+1} y^0 + \sum_{k=1}^{n} \binom{n}{k} x^k y^{n+1-k} + \binom{n}{0} x^0 y^{n+1} \\ &= \binom{n}{0} x^0 y^{n+1} + \sum_{k=1}^{n} \left(\binom{n}{k-1} x^k y^{n+1-k} + \binom{n}{k} x^k y^{n+1-k} \right) + \binom{n}{n} x^{n+1} y^0. \end{aligned}$$

We utilize the well-known identities

$$\binom{n}{0} = \binom{n+1}{0}, \quad \binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}, \quad \binom{n}{n} = \binom{n+1}{n+1}.$$

For the second identity here, we recall that $k-1 \geq 0$ since $k \geq 1$ and that $k \leq n$, so the binomial coefficients are indeed well-defined. This gives

$$\begin{aligned}
&\sum_{k=1}^{n+1} \binom{n}{k-1} x^k y^{n+1-k} + \sum_{k=0}^{n} \binom{n}{k} x^k y^{n+1-k} \\
&\quad = \binom{n+1}{0} x^0 y^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} x^k y^{n+1-k} + \binom{n+1}{n+1} x^{n+1} y^0 \\
&\quad = \sum_{k=0}^{n+1} \binom{n+1}{k} x^k y^{n+1-k}.
\end{aligned}$$

We conclude from the assumption that the identity holds for $n \in \mathbb{N}$ we also get that the identity holds for $n+1$.

By the principle of induction, the identity holds for all $n \in \mathbb{N}$, and thus the proof is complete. □

Corollary 1
For all $n \in \mathbb{N}$ we have

$$2^n = \sum_{k=0}^{n} \binom{n}{k}.$$

Proof. We use the Binomial Theorem with $x = y = 1$. This gives

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}.$$

□

Corollary 2
Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Then

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Proof. We use the Binomial Theorem with x as in the statement and $y = 1$. This gives

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

□

Remark 1
The intution for the binomial theorem is as follows: when we expand the n-th power of $x+y$,

$$(x+y)^5 = (x+y)(x+y)(x+y)(x+y)(x+y),$$

and use the distributive law, then we obtain products of x and y in different combinations

$$(x+y)^5 = xxxxx + xxxxy + xxxyx + xxyxx + \ldots$$

Those terms where the factors x and y appear in equal numbers can be summarized, and its easy to see that there are $\binom{n}{k}$ terms that contain the factor x to the k-th power and the factor y to the $(n-k)$-th power.

$$xxxxx = \binom{5}{0} x^5 y^0,$$

$$xxxxy + xxxyx + xxyxx + xyxxx + yxxxx = \binom{5}{1} x^4 y^1,$$

$$\dots$$

This line of thought gives an less formal proof for the binomial theorem. Even though most mathematicians will accept such a reasoning as a proof, the more formal proof above contains many techniques that come in helpful when the result is not as obvious.