MATH 109 – EXERCISES

To be discussed within the first week.

Exercise 1. Let a, b be real numbers with $b \neq 0$ and $b+1 \neq 0$. Under which condition on a and b do we have

$$\frac{1+a}{1+b} < \frac{a}{b}$$

Solution 1. Let a and b as in the statement of the problem. Since b and $b+1$ are non-zero, they are positive or negative numbers. We make a case distinction.

If b and $b+1$ have the same sign (i.e., are both positive or both negative) then

$$\frac{1+a}{1+b} < \frac{a}{b} \iff (1+a)b < a(1+b) \iff b + ab < a + ab \iff b < a.$$

If b and $b+1$ have different signs, then

$$\frac{1+a}{1+b} < \frac{a}{b} \iff (1+a)b > a(1+b) \iff b + ab > a + ab \iff b > a.$$

The conclusion is that if b and $b+1$ have the same sign, then the inequality holds if and only if $b < a$, and the if b and $b+1$ have different signs, then the inequality holds if and only if $b > a$. □

Exercise 2. Let a, b, c be real numbers with $b \neq 0$ and $b+c \neq 0$. Under which condition on a, b and c do we have

$$\frac{c+a}{c+b} < \frac{a}{b}$$

Solution 2. *This is a modification of the previous proof.*

Let a and b as in the statement of the problem. Since b and $b+c$ are non-zero, they are positive or negative numbers. We make a case distinction.

If b and $b+c$ have the same sign (i.e., are both positive or both negative) then

$$\frac{c+a}{c+b} < \frac{a}{b} \iff (c+a)b < a(c+b) \iff bc + ab < ac + ab \iff bc < ac.$$

If b and $b+c$ have different signs, then

$$\frac{c+a}{c+b} < \frac{a}{b} \iff (c+a)b > a(c+b) \iff bc + ab > ac + ab \iff bc > ac.$$

The conclusion is that if b and $b+1$ have the same sign, then the inequality holds if and only if $bc < ac$, and the if b and $b+1$ have different signs, then the inequality holds if and only if $bc > ac$. □

Exercise 3. Proof that for all non-negative real numbers a, b we have

$$\sqrt{ab} \leq \frac{a}{2} + \frac{b}{2}$$

Solution 3. Multiplying both sides of the equation by 2 and bringing $\sqrt{ab}$ to the right hand side, we get that the equation is equivalent to

$$0 \leq a - 2\sqrt{ab} + b.$$

Since the latter equals $(\sqrt{a} - \sqrt{b})^2$, and a square is always nonnegative, the exercise follows. □

Exercise 4. Show that for all non-negative real numbers a, b, c we have

$$\sqrt[3]{abc} \leq \frac{a+b+c}{3}$$

Solution 4. Write $x = \sqrt[3]{a}, y = \sqrt[3]{b}, z = \sqrt[3]{c}$. Multiplying by 3, the equation then becomes

$$3xyz \leq x^3 + y^3 + z^3 \quad \text{or equivalently} \quad 0 \leq x^3 + y^3 + z^3 - 3xyz$$

Now we note that

$$\begin{aligned} x^3 + y^3 + z^3 - 3xyz &= (x + y + z)(x^2 + y^2 + z^2 - xy - zx - yz) \\ &= (x + y + z) \cdot \frac{1}{2} \cdot (x^2 - 2xy + y^2 + x^2 - 2zx + z^2 + y^2 - 2yz + z^2) \\ &= (x + y + z) \cdot \frac{1}{2} \cdot ((x - y)^2 + (x - z)^2 + (y - z)^2). \end{aligned}$$

Since x, y, z are positive, the latter equation is positive and the solution follows. □

Exercise 5. Proof that $7^{\log_2(2t)} = 7t^{\log_2(7)}$ for all positive real numbers $t > 0$.

Solution 5. Using the laws of computing with logarithms, we have

$$\begin{aligned} 7^{\log_2(2t)} = 7^{\log_2(2)+\log_2(t)} = 7^{1+\log_2(t)} = 7 \cdot 7^{\log_2(t)} = 7 \cdot 2^{\log_2(7^{\log_2(t)})} \\ = 7 \cdot 2^{\log_2(t)\cdot\log_2(7)} = 7 \cdot 2^{\log_2(t^{\log_2(7)})} = 7 \cdot t^{\log_2(7)}. \end{aligned}$$

This had to be shown. □

Exercise 6. Consider a cubic polynomial

$$f(x) = ax^3 + bx^2 + cx + d$$

where a, b, c, d are real numbers. Suppose that x_0 is a real number that satisfies $f(x_0) = 0$. Find real numbers p, q, r, t such that the quadratic polynomial

$$g(x) = px^2 + qx + r$$

satisfies

$$f(x) = (x - t)q(x).$$

Express p, q, r, t in terms of a, b, c, d, x_0.

Solution 6. This is exactly polynomial long division. □

Exercise 7. Find the roots of the cubic polynomial

$$p(x) = 2x^3 - 10x^2 - 314x + 2002.$$

Explain how you found the answer.

Solution 7. Obviuously, $x \in \mathbb{R}$ is a root of p if and only if it is a root of $1/2p$. Hence we can equivalently search for the roots of

$$1/2p(x) = x^3 - 5x^2 - 157x + 1001.$$

It is easily seen that $1001 = 700 + 280 + 21$ is divisible by 7, and we get $1001/7 = 100 + 40 + 3 = 143$. The latter is obviously divisible by 11, and we have $143/11 = 13$.

We guess that p has at least one integer root. Indeed, one computes that $p(7) = 0$. We also check that $p(11) = 0$ and $p(13) = 0$.

Since a cubic function assumes the value zero at most at three different places, this gives all roots of the polynomial. □

Exercise 8. For which real numbers x, y do we have

$$6x^2 + 5y^2 + 13xy + 17x + 12y + 7 = 0$$

Solution 8. Trying to factor $6x^2 + 5y^2 + 13xy + 17x + 12y + 7 = (ax + by + c)(dx + ey + f)$, one sees that we can write

$$6x^2 + 5y^2 + 13xy + 17x + 12y + 7 = (2x + y + 1)(3x + 5y + 7).$$

Hence the solution to the given equation boils down to solving

$$2x + y + 1 = 0 \quad \text{and} \quad 3x + 5y + 7 = 0.$$

In other words, the real numbers x, y satisfying the equation are equal to the union of the two lines $y = -2x - 1$ and $y = \frac{-3}{5}x - \frac{-7}{5}$. $\square$