MATH 109 – HOMEWORK 2

Due Friday 26th. Handwritten submissions only.
The exercises in this homework are worth 16 points.

Exercise 1
A logical proposition that is composed from statements $A, B, C, \ldots$ through a combination of negation, disjunction, and conjunction is called in *conjunctive normal form* if it is the conjunction of disjunctions of terms from $A, \neg A, B, \neg B, \ldots$.

For example, the following proposition is in conjunctive normal form:

$$(C \vee \neg A \vee \neg B) \wedge (\neg B \vee A) \wedge (\neg C \vee B \vee A).$$

For each of the following propositions, find an equivalent proposition in conjunctive normal form:

- $((A \vee B) \wedge \neg B) \vee (B \wedge \neg(A \vee C) \wedge \neg(A \vee B)) \vee (A \wedge B)$
- $(\neg B \wedge A) \vee (\neg A) \vee (\neg C \wedge \neg(A \vee B))$
- $\neg((A \vee \neg C) \wedge (C \wedge B) \wedge \neg(A \vee B \vee \neg C))$

Solution 1
For the first one, we check that

$$\begin{aligned}
&((A \vee B) \wedge \neg B) \vee (B \wedge \neg(A \vee C) \wedge \neg(A \vee B)) \vee (A \wedge B) \\
\iff &((A \wedge \neg B) \vee (B \wedge \neg B)) \vee (B \wedge \neg A \wedge \neg C \wedge \neg A \wedge \neg B) \vee (A \wedge B) \\
\iff &((A \wedge \neg B) \vee (B \wedge \neg B)) \vee F \vee (A \wedge B) \\
\iff &((A \wedge \neg B) \vee (B \wedge \neg B)) \vee (A \wedge B) \\
\iff &((A \wedge \neg B) \vee F) \vee (A \wedge B) \\
\iff &(A \wedge \neg B) \vee (A \wedge B) \\
\iff &A \wedge (B \vee \neg B) \\
\iff &A
\end{aligned}$$

Second, we observe

$$\begin{aligned}
&(\neg B \wedge A) \vee (\neg A) \vee (\neg C \wedge \neg(A \vee B)) \\
\iff &(\neg B \wedge A) \vee (\neg A) \vee (\neg C \wedge \neg A \wedge \neg B) \\
\iff &((\neg B \vee \neg A) \wedge (A \vee \neg A)) \vee (\neg C \wedge \neg A \wedge \neg B) \\
\iff &((\neg B \vee \neg A) \wedge T) \vee (\neg C \wedge \neg A \wedge \neg B) \\
\iff &(\neg B \vee \neg A) \vee (\neg C \wedge \neg A \wedge \neg B) \\
\iff &(\neg B \vee \neg A) \vee (\neg C \wedge \neg A \wedge \neg B) \\
\iff &((\neg B \vee \neg A) \vee \neg C) \wedge ((\neg B \vee \neg A) \vee \neg A) \wedge ((\neg B \vee \neg A) \vee \neg B) \\
\iff &(\neg B \vee \neg A) \vee (\neg C \wedge \neg A \wedge \neg B) \\
\iff &(\neg B \vee \neg A \vee \neg C) \wedge (\neg B \vee \neg A \vee \neg A) \wedge (\neg B \vee \neg A \vee \neg B) \\
\iff &(\neg B \vee \neg A \vee \neg C) \wedge (\neg B \vee \neg A) \wedge (\neg B \vee \neg A) \\
\iff &(\neg B \vee \neg A \vee \neg C) \wedge (\neg B \vee \neg A)
\end{aligned}$$

Third, we observe

$$\begin{aligned}
&\neg((A\vee C)\wedge(C\wedge B)\wedge\neg(A\vee B\vee\neg C))\\
\iff&\neg(A\vee C)\vee\neg(C\wedge B)\vee(A\vee B\vee\neg C)\\
\iff&\neg(A\vee C)\vee\neg(C\wedge B)\vee A\vee B\vee\neg C\\
\iff&\neg(A\vee C)\vee\neg C\vee\neg B\vee A\vee B\vee\neg C\\
\iff&\neg(A\vee C)\vee\neg C\vee T\vee A\\
\iff&T
\end{aligned}$$

Exercise 2
Assume that we have parametrized statements $X(a,b)$ and $Y(a)$ that satisfy

$$X(a,b)\iff Y(a)\wedge Y(b).$$

Show that the following equivalence holds:

$$(X(a,b)\wedge Y(c))\iff\neg(\neg Y(a)\vee\neg X(b,c)).$$

Solution 2
We observe that

$$(X(a,b)\wedge Y(c))\iff(Y(a)\wedge Y(b)\wedge Y(c))\iff(Y(a)\wedge X(b,c))\iff\neg(\neg Y(a)\vee\neg X(b,c)).$$

Exercise 3
Let x,y,z be three irrational numbers. Show that there are two of them whose sum is again irrational.

Solution 3
Suppose that this is not the case. Then there exist x,y,z being irrational numbers such that the sums $x+y$, $y+z$ and $z+x$ are rational numbers. Observe that

$$x=\frac{x+x}{2}=\frac{x+z-z-y+y+x}{2}=\frac{x+z}{2}-\frac{z+y}{2}+\frac{y+x}{2}.$$

Since the sums $x+y$, $y+z$, and $x+z$ are rational, so are their halves, and hence is the sum of the latter. It follows that x is a rational number, which contradicts x being irrational.

Hence the one of the sums $x+y$, $y+z$, and $x+z$ must be irrational.

Exercise 4
Find all the pairs of non-zero real numbers (x,y) which satisfy

$$x+\frac{x}{y}=\frac{8}{3},\qquad y+\frac{1}{x}=\frac{5}{2}.$$

Solution 4
Suppose that x and y are non-zero and satisfy the two equations. We observe that

$$xy+x=\frac{8y}{3},\quad xy+1=\frac{5x}{2}.$$

Hence

$$\begin{aligned}\frac{8}{3}y - x = \frac{5}{2}x - 1 &\iff \frac{8}{3}y = \frac{7}{2}x - 1\\ &\iff y = \frac{21}{16}x - \frac{3}{8}\\ &\iff \frac{21}{16}x - \frac{3}{8} + \frac{1}{x} = \frac{5}{2}\\ &\iff \frac{21}{16}x + \frac{1}{x} = \frac{23}{8}.\end{aligned}$$

Again using that x is non-zero, we find

$$\begin{aligned}\frac{21}{16}x + \frac{1}{x} = \frac{23}{8} &\iff \frac{21}{16}x^2 + 1 = \frac{23}{8}x\\ &\iff x^2 + \frac{16}{21} = \frac{16 \cdot 23}{21 \cdot 8}x\\ &\iff x^2 + \frac{16}{21} = \frac{64}{21}x\end{aligned}$$

Hence to find the solution we have to find x such that

$$x^2 - \frac{64}{21}x + \frac{16}{21} = 0.$$

Standard techniques for quadratic equations apply.

Exercise 5
Find all real numbers x that satisfy the equation

$$8^x + 2 = 4^x + 2^{x+1}.$$

Solution 5
Assume that x satisfies this equation. We observe that

$$8^x + 2 = (2^x)^3 + 2, \quad 4^x + 2^{x+1} = (2^x)^2 + 2 \cdot 2^x$$

Define $y = 2^x$. From assumptions we get

$$y^3 - y^2 - 2y + 2 = y^2(y-1) - 2(y-1) = (y^2 - 2)(y-1) = 0.$$

We conclude that $y = 1$ or $y = \sqrt{2}$ or $y = -\sqrt{2}$. But by construction y is positive, and thus $y = 1$ or $y = \sqrt{2}$ must hold. Now $y = 2^x$ implies

$$x = \log_2(1) = 0 \text{ or } x = \log_2 \sqrt{2} = \frac{1}{2}\log_2(2) = \frac{1}{2}.$$

Checking both possible values then verifies that 0 and $\frac{1}{2}$ are exactly the solutions.

Exercise 6
Let a be an odd number. Show that there exists an integer k such that $a^2 = 8k + 1$.

Solution 6
Let a be a odd, so $a = 2b + 1$ for some integer b. Hence

$$a^2 = 4b^2 + 4b + 1 = 4b(b+1) + 1.$$

Let k be the real number such that $a^2 = 8k + 1$. Then

$$4b(b+1) + 1 = 8k + 1,$$

and hence

$$b(b+1)/2 = k.$$

Since $b(b+1)$ is an integer and even, we now see that k is an integer.