MATH 109 – HOMEWORK 4

Due Friday, February 9th. Handwritten submissions only.
The exercises in this homework are worth 16 points.

Exercise 1
Let $n \in \mathbb{N}_0$ and $p, q \in \mathbb{N}_0$ with $p \leq n/2$ and $q \leq n/2$. Prove that

$$\binom{n}{p}\binom{n-p}{q} = \binom{n}{q}\binom{n-q}{p}.$$

Solution 1
One strategy to prove the equality of two things is prove that both of them are equal to a third thing.
Unfolding definitions, we get

$$\begin{aligned}\binom{n}{p}\binom{n-p}{q} &= \frac{n!}{p!(n-p)!}\frac{(n-p)!}{q!(n-p-q)!} = \frac{n!}{p!q!(n-p-q)!}\\ \binom{n}{q}\binom{n-q}{p} &= \frac{n!}{q!(n-q)!}\frac{(n-q)!}{p!(n-q-p)!} = \frac{n!}{p!q!(n-p-q)!},\end{aligned}$$

which proves the result.

Exercise 2
Simplify the following expressions as much as possible (without computing the sum completely):

$$a := k\sum_{k=1}^{100}\sqrt{k}, \quad b := \sum_{i=1}^{100} i^2\log(i) + \sum_{i=101}^{200} i^2\log(i), \quad c := \sum_{l=25}^{5} l^{\pi}$$
$$d := \sum_{t=0}^{50}\ln(t+1) + \sum_{t=0}^{50}\ln\left(\frac{1}{50-t+1}\right)$$

Solution 2
For the term a, we have to rename the index variable. For the term b, we use that the ranges of the index variable can be concatenated. For the term c, we observe that the upper index bound is smaller than the lower index bound, hence the entire term is zero. For the term d, we note that the second sum term can be

reordered; we then use properties of the natural logarithm and the distributive law.

$$
\begin{aligned}
a &= k\sum_{k=1}^{100}\sqrt{k} = k\sum_{l=1}^{100}\sqrt{l} = \sum_{l=1}^{100}k\sqrt{l},\\
b &= \sum_{i=1}^{100} i^2\log(i) + \sum_{i=101}^{200} i^2\log(i), = \sum_{i=1}^{200} i^2\log(i),\\
c &= \sum_{l=25}^{5} l^{\pi} = 0\\
d &= \sum_{t=0}^{50}\ln(t+1) + \sum_{t=0}^{50}\ln\left(\frac{1}{50-t+1}\right) = \sum_{t=0}^{50}\ln(t+1) + \sum_{t=0}^{50}\ln\left(\frac{1}{t+1}\right)\\
&= \sum_{t=0}^{50}\ln(t+1) - \sum_{t=0}^{50}\ln(t+1) = \sum_{t=0}^{50}\ln(t+1) - \ln(t+1) = 0
\end{aligned}
$$

Exercise 3

Let A, B, C be sets. Proof the following statements.

- We have $A \subseteq B$ if and only if $A \cup B = B$.
- We have $A \subseteq B$ if and only if $A \cap B = A$.
- We have $A \subseteq B$ if and only if $A \setminus B = \emptyset$.

Solution 3

The proofs for these statements use the following ideas: to show the equivalences of two statements, it suffices to show that they imply each other. Moreover, to show that two sets X and Y are the same, it suffices to show that $X \subseteq Y$ and $Y \subseteq X$.

- Suppose that $A \subseteq B$. Then every element of A is an element of B. We have $B \subseteq A \cup B$ by definition of the union, and we have

  $$x \in A \cup B \iff x \in A \vee x \in B \implies x \in B \vee x \in B \iff x \in B,$$

  so every element of $A \cup B$ is an element of B. Hence B and $A \cup B$ are mutual subsets of each other, we get that $A \cup B = B$.

  Suppose in turn that $A \cup B = B$. We then find that

  $$x \in A \implies x \in A \cup B \iff x \in A \vee x \in B \iff x \in B,$$

  so every element of A is an element of B as well.

  This shows that $A \subseteq B$ and $A \cup B = B$ imply each other, and are hence equivalent.
- Suppose that $A \subseteq B$. Then we have $A \cap B \subseteq A$, and thus, to show that $A \cap B = A$, it remains to show that $A \subseteq A \cap B$. Now, since $A \subseteq B$, we already have

  $$x \in A \implies x \in A \wedge x \in B \implies x \in A \cap B.$$

  Hence $A \subseteq B$ implies that $A = A \cap B$.

  In turn, suppose that $A = A \cap B$. If $x \in A$, then we have $x \in A$ and $x \in B$, which in particular implies/includes that $x \in B$. Hence $A = A \cap B$ implies $A \subseteq B$.

  Since $A \subseteq B$ and $A \cap B = A$ imply each other, we conclude that both statements are equivalent.
- Suppose that $A \subseteq B$. Then $A \setminus B$ contains those elements of A that are not contained in B, but since all elements in A are already contained in B, we conclude that there are no elements of A not contained in B, and thus $A \setminus B$ is the empty set.

  In turn, if $A \setminus B = \emptyset$, then by definition there does not exist a member of A that is not contained in B. But that is equivalent to saying that all members of A are contained in B, which is just the definition of the statement $A \subseteq B$.

  Thus $A \subseteq B$ and $A \setminus B = \emptyset$ imply each other, and are equivalent statements hence.

Exercise 4
Let A, B, C, D be sets. Proof the following implications. For each implication, give a counterexample why it is not an equivalence.

- If $C \subseteq A$ and $D \subseteq B$, then $C \cup D \subseteq A \cup B$.
- If $C \subseteq A$ and $D \subseteq B$, then $C \cap D \subseteq A \cap B$.
- If $A \subseteq B$, then $C \setminus B \subseteq C \setminus A$.

Solution 4

- Suppose that $C \subseteq A$ and $D \subseteq B$, i.e., we have impliciations
$$x \in C \implies x \in A, \quad x \in D \implies x \in B.$$
We then find that
$$x \in C \cup D \iff x \in C \vee x \in D \implies x \in A \vee x \in B \iff x \in A \cup B.$$
This shows the desired implication. However, the converse implication does not hold: there exist sets A, B, C, D such that $C \cup D \subseteq A \cup B$ but we do not have $C \subseteq A$ or $D \subseteq B$. For example,
$$A = \{1, 2\}, \quad B = \{3, 4\}, \quad C = \{1, 3\}, \quad A = \{2, 4\}.$$
- Suppose that $C \subseteq A$ and $D \subseteq B$, i.e., we have impliciations
$$x \in C \implies x \in A, \quad x \in D \implies x \in B.$$
We then find that
$$x \in C \cap D \iff x \in C \wedge x \in D \implies x \in A \wedge x \in B \iff x \in A \cap B.$$
This shows the desired implication. However, the converse implication does not hold: there exist sets A, B, C, D such that $C \cap D \subseteq A \cap B$ but we do not have $C \subseteq A$ or $D \subseteq B$. We can just the same example as in the previous subproblem.
- Suppose that $A \subseteq B$, i.e., we have $x \in A$ implying $x \in B$. This implication gives us also that
$$x \notin B \implies x \notin A.$$
For any set C we now find that
$$x \in C \setminus B \iff x \in C \wedge x \notin B \implies x \in C \wedge x \notin A \iff x \in C \setminus A.$$
Hence $A \subseteq B$ implies $C \setminus B \subseteq C \setminus A$. The converse implication, however, is generally false. For example, consider the intervals
$$A = [0, 2], \quad B = [0, 1], \quad C = [-1, 1].$$

Exercise 5
Recall the definition of the factorial $n!$ for $n \in \mathbb{N}_0$:
$$n! := \prod_{k=1}^{n} k$$

(a) Write down the values 0!, 1!, 2!, and 3!.
(b) Write the following expressions in terms of one product:
$$a_{n,k} := n!/k!, \quad b_n := (n!)^2, \quad c_n := \sum_{k=1}^{n} \ln(k).$$

Solution 5
We have
$$0! = 1, \quad 1! = 1, \quad 2! = 2, \quad 3! = 6.$$

We observe

$$a_{n,k} = n!/k! = \prod_{m=k+1}^{n} m, \quad b_n = n!n! = \prod_{m=1}^{n} m^2, \quad c_n = \sum_{k=1}^{n} \ln(k) = \ln\left(\prod_{k=1}^{n} k\right).$$

Exercise 6
Prove that the square root function and the logarithm function are concave, i.e.,

(1) Prove that for all $x, y \in \mathbb{R}^+$ we have

$$\frac{1}{2}\sqrt{x} + \frac{1}{2}\sqrt{y} \leq \sqrt{\frac{1}{2}x + \frac{1}{2}y}.$$

(2) Prove that for all $x, y \in \mathbb{R}^+$ we have

$$\frac{1}{2}\ln(x) + \frac{1}{2}\ln(y) \leq \ln\left(\frac{1}{2}x + \frac{1}{2}y\right).$$

Solution 6
One approach to proving an inequality is to reduce to an inequality that is already known.

Let $x, y \in \mathbb{R}^+$. We first prove the inequality for the square root function. We have

$$\begin{aligned}
\frac{1}{2}\sqrt{x} + \frac{1}{2}\sqrt{y} \leq \sqrt{\frac{1}{2}x + \frac{1}{2}y} \quad &\iff \quad \frac{1}{4}x + \frac{1}{2}\sqrt{xy} + \frac{1}{4}y \leq \frac{1}{2}x + \frac{1}{2}y \\
&\iff \quad \sqrt{xy} \leq \frac{1}{2}x + \frac{1}{2}y \\
&\iff \quad xy \leq \frac{1}{4}x^2 + \frac{1}{2}xy + \frac{1}{4}y^2 \\
&\iff \quad xy \leq \frac{1}{2}x^2 + \frac{1}{2}y^2
\end{aligned}$$

The preceding inequality (Young's inequality) has been proven earlier in this course.

As for the inequality, concerning the natural logarithm, we use standard laws for the logarithm function to find

$$\begin{aligned}
\frac{1}{2}\ln(x) + \frac{1}{2}\ln(y) \leq \ln\left(\frac{1}{2}x + \frac{1}{2}y\right) \quad &\iff \quad \ln\left(\sqrt{xy}\right) \leq \ln\left(\frac{1}{2}x + \frac{1}{2}y\right) \\
&\iff \quad 0 \leq \ln\left(\frac{1}{2}x + \frac{1}{2}y\right) - \ln\left(\sqrt{xy}\right) \\
&\iff \quad 0 \leq \ln\left(\frac{\frac{1}{2}x + \frac{1}{2}y}{\sqrt{xy}}\right) \\
&\iff \quad 1 \leq \frac{\frac{1}{2}x + \frac{1}{2}y}{\sqrt{xy}}
\end{aligned}$$

The last inequality follows again from Young's inequality, proven earlier in this course.