MATH 109 – HOMEWORK 4

Due Friday, February 9th. Handwritten submissions only.
The exercises in this homework are worth 16 points.

Exercise 1
Let $n \in \mathbb{N}_0$ and $p, q \in \mathbb{N}_0$ with $p \leq n/2$ and $q \leq n/2$. Prove that

$$\binom{n}{p}\binom{n-p}{q} = \binom{n}{q}\binom{n-q}{p}.$$

Exercise 2
Simplify the following expressions as much as possible (without computing the sum completely):

$$a := k\sum_{k=1}^{100} \sqrt{k}, \quad b := \sum_{i=1}^{100} i^2 \log(i) + \sum_{i=101}^{200} i^2 \log(i), \quad c := \sum_{l=25}^{5} l^\pi$$

$$d := \sum_{t=0}^{50} \ln(t+1) + \sum_{t=0}^{50} \ln\left(\frac{1}{50-t+1}\right)$$

Exercise 3
Let A, B, C be sets. Proof the following statements.

- We have $A \subseteq B$ if and only if $A \cup B = B$.
- We have $A \subseteq B$ if and only if $A \cap B = A$.
- We have $A \subseteq B$ if and only if $A \setminus B = \emptyset$.

Exercise 4
Let A, B, C, D be sets. Proof the following implications. For each implication, give a counterexample why it is not an equivalence.

- If $C \subseteq A$ and $D \subseteq B$, then $C \cup D \subseteq A \cup B$.
- If $C \subseteq A$ and $D \subseteq B$, then $C \cap D \subseteq A \cap B$.
- If $A \subseteq B$, then $C \setminus B \subseteq C \setminus A$.

Exercise 5
Recall the definition of the factorial $n!$ for $n \in \mathbb{N}_0$:

$$n! := \prod_{k=1}^{n} k$$

(a) Write down the values 0!, 1!, 2!, and 3!.
(b) Write the following expressions in terms of one product:

$$a_{n,k} := n!/k!, \quad b_n := (n!)^2, \quad c_n := \sum_{k=1}^{n} \ln(k).$$

Exercise 6
Prove that the square root function and the logarithm function are concave, i.e.,

(1) Prove that for all $x, y \in \mathbb{R}^+$ we have

$$\frac{1}{2}\sqrt{x} + \frac{1}{2}\sqrt{y} \leq \sqrt{\frac{1}{2}x + \frac{1}{2}y}.$$

(2) Prove that for all $x, y \in \mathbb{R}^+$ we have

$$\frac{1}{2}\ln(x) + \frac{1}{2}\ln(y) \leq \ln\left(\frac{1}{2}x + \frac{1}{2}y\right).$$