MATH 109 – HOMEWORK 6

Due Friday, February 23rd. Handwritten submissions only.
The exercises in this homework are worth 16 points.

Problem 1
Let $n \in \mathbb{N}$. Prove using the principle of induction:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

Solution 1
We prove both identities by the principle of induction.

- Consider the identity

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

First, we check that for $n = 1$ this statement is true. This is simple:

$$\sum_{k=1}^{1} k^2 = 1^2 = 1 = \frac{1(1+1)(2+1)}{6},$$

Second, we conduct the induction step: assuming that the statement is true for some $n \in \mathbb{N}$, we want to show that it is true for $n+1$. We observe

$$\begin{aligned}\sum_{k=1}^{n+1} k^2 &= \sum_{k=1}^{n} k^2 + (n+1)^2 \\ &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2,\end{aligned}$$

where we have used the induction hypothesis. Now we simplify:

$$\begin{aligned}\frac{n(n+1)(2n+1)}{6} + (n+1)^2 &= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} \\ &= \frac{(n^2+n)(2n+1) + 6(n^2+2n+1)}{6} \\ &= \frac{2n^3 + 2n^2 + n^2 + n + 6n^2 + 12n + 6}{6} \\ &= \frac{2n^3 + 9n^2 + +13n + 6}{6}\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{(n+1)(n+2)(2(n+1)+1)}{6} = \frac{(n+1)(n+2)(2n+3)}{6} &= \frac{(n^2+3n+2)(2n+3)}{6} \\ &= \frac{2n^3 + 6n^2 + 4n + 3n^2 + 9n + 6}{6} \\ &= \frac{2n^3 + 9n^2 + 13n + 6}{6}.\end{aligned}$$

This completes the induction step.

We have proven that the identity is true for $n = 1$, and that the identity being true some $n \in \mathbb{N}$ implies the identity being true for $n+1$ too. By the principle of induction, the identity holds for all $n \in \mathbb{N}$. The proof is complete.

- Consider the identity

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

First, we check that for $n = 1$ this statement is true, which is not difficult:

$$\sum_{k=1}^{1} k^3 = 1^3 = 1 = \frac{1^2(1+1)^2}{4}.$$

Second, we conduct the induction step: assuming that the statement is true for some $n \in \mathbb{N}$, we want to show that it is true for $n+1$. We observe

$$\begin{aligned}\sum_{k=1}^{n+1} k^3 &= \sum_{k=1}^{n} k^3 + (n+1)^3 \\ &= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\ &= \frac{(n^2+4n+4)(n+1)^2}{4},\end{aligned}$$

where we have used the induction hypothesis and applied some simple algebraic manipulation in the last step. On the other hand, we have

$$\frac{(n+1)^2(n+2)^2}{4} = \frac{(n+1)^2(n^2+4n+4)}{4}.$$

This completes the induction step.

We have proven that the identity is true for $n = 1$, and that the identity being true some $n \in \mathbb{N}$ implies the identity being true for $n+1$ too. By the principle of induction, the identity holds for all $n \in \mathbb{N}$. The proof is complete.

Problem 2

Let $n \in \mathbb{N}$ with $n \geq 4$. Prove that $n!$ is divisible by a square number $s \in \mathbb{N}$ with $\sqrt{s} \geq \lfloor n/2 \rfloor$.

Solution 2

One may get an idea why this is true when writing out the factorials for a few small numbers.

First, we consider the case that n is even. Then $n = 2k$ for some $k \in \mathbb{N}$, and we have $k = n/2 = \lfloor n/2 \rfloor$. In particular,

$$n! = n(n-1) \cdot \dots \cdot 2 \cdot 1 = (2k)(2k-1) \cdot \dots \cdot (k+1)(k)(k-1) \cdot \dots \cdot 2 \cdot 1.$$

Obviously, $n!$ is divisible by k^2. Hence $n!$ is divisible by the square number $s = k^2$, and

$$\sqrt{s} = k \geq n/2 = \lfloor n/2 \rfloor.$$

Second, consider the case that n is odd. Then $n-1$ is even. We have that $n-1$ is divisible by a square number s such that $\sqrt{s} \geq (n-1)/2 = \lfloor (n-1)/2 \rfloor$. Then $n!$ is divisible by s too, and $\sqrt{s} \geq \lfloor (n-1)/2 \rfloor = \lfloor n/2 \rfloor$. Hence there exists a square number as desired for $n \geq 4$ odd.

The truth of the statement follows.

Problem 3

Let $p \in \mathbb{N}_0$ and $n \in \mathbb{N}$. Prove Pascal's identity:

$$\sum_{l=1}^{p+1} \binom{p+1}{l} S_{n,p+1-l} = (n+1)^{p+1} - 1, \qquad \text{where} \qquad S_{n,p} := \sum_{k=1}^{n} k^p.$$

Use this formula to compute $S_{n,4}$ for arbitrary $n \in \mathbb{N}$.

Solution 3
We observe that

$$\begin{aligned}\sum_{l=1}^{p+1}\binom{p+1}{l}S_{n,p+1-l} &= \sum_{l=1}^{p+1}\binom{p+1}{l}\sum_{k=1}^{n}k^{p+1-l}\\ &= \sum_{k=1}^{n}\sum_{l=1}^{p+1}\binom{p+1}{l}k^{p+1-l}\\ &= \sum_{k=1}^{n}\left(\sum_{l=0}^{p+1}\binom{p+1}{l}k^{p+1-l}-k^{p+1}\right)\\ &= \sum_{k=1}^{n}\left((k+1)^{p+1}-k^{p+1}\right) = (n+1)^{p+1}-1.\end{aligned}$$

Finally, we obtain a formula for $S_{n,4}$. As a preparation to make this more comfortable, we first conduct the general simplification

$$\sum_{l=1}^{p+1}\binom{p+1}{l}S_{n,p+1-l} = \sum_{l=1}^{p+1}\binom{p+1}{l}S_{n,p-(l-1)} = \sum_{l=0}^{p}\binom{p+1}{l+1}S_{n,p-l}.$$

Now in the case $p=4$ we get

$$\begin{aligned}\sum_{l=1}^{p+1}\binom{p+1}{l}S_{n,p+1-l} &= \sum_{l=0}^{4}\binom{4+1}{l+1}S_{n,4-l}\\ &= \binom{4+1}{0+1}S_{n,4-0}+\binom{4+1}{1+1}S_{n,4-1}+\binom{4+1}{2+1}S_{n,4-2}+\binom{4+1}{3+1}S_{n,4-3}+\binom{4+1}{4+1}S_{n,4-4}\\ &= \binom{5}{1}S_{n,4}+\binom{5}{2}S_{n,3}+\binom{5}{3}S_{n,2}+\binom{5}{4}S_{n,1}+\binom{5}{5}S_{n,0}\\ &= 5S_{n,4}+10S_{n,3}+10S_{n,2}+5S_{n,1}+S_{n,0}\end{aligned}$$

For the terms $S_{n,0}, S_{n,1}, S_{n,2}, S_{n,3}$, we have already seen formulas in the homework and/or the lectures. Using Pascal's identity, we thus can derive

$$5S_{n,4} = (n+1)^5 - 1 - 10S_{n,3} - 10S_{n,2} - 5S_{n,1} - S_{n,0}.$$

Plugging formulas for the terms on the right-hand side and then dividing by 5, we are done.

Problem 4
Find the mistake in the following reasoning:

Claim: All cars have the same color.

We use the principle of induction: we show that for all $n \in \mathbb{N}$ within every set of n cars all cars have the same color.

First, if $n = 1$, then certainly every set containing 1 car has all cars with the same color.

Second, suppose the for all sets of n cars we have the cars in that set having the same color. Consider now a set $A = \{a_1, \dots, a_n, a_{n+1}\}$ of $n+1$ cars, and define

$$A_1 = \{a_2, \dots, a_n, a_{n+1}\}, \quad A_{n+1} = \{a_1, \dots, a_n\}.$$

By the induction assumption, we have that all cars in A_1 and all cars in A_{n+1} have the same color. Hence all cars in $A = A_1 \cup A_{n+1}$ have the same color.

Solution 4
The induction step is conducted incorrectly for $n = 1$, when proving the statement for $n = 2$.

To illustrate the mistake, we scrutinize the induction step. The reasoning in the induction step is as follows: Having a set of $n+1$ cars,

$$A = \{a_1, \dots, a_n, a_{n+1}\}$$

we split up A into two sets

$$A_1 = \{a_2, \dots, a_n, a_{n+1}\}, \quad A_{n+1} = \{a_1, \dots, a_n\}.$$

By the induction assumption, the cars in the sets A_1 and A_{n+1} have the same color. Suppose we pick a car a_i with $1 < i < n+1$. Then this car a_i has the same color as the car a_1 and the same color as the car a_{n+1}, which implies that the cars a_1 and a_{n+1} have the same color too. In particular, one may conclude that all the cars in the set A share the same color.

The problem is that in the case $n = 1$ there is no index i that satisfies $1 < i < 2$. In other words, if $A = \{a_1, a_2\}$, then the sets $A_1 = \{a_1\}$ and $A_2 = \{a_2\}$ have no car in common. This is where the induction step fails.

If the claim would be true $n = 2$, then indeed we could prove that all cars have the same color. The induction step does not work when going from $n = 1$ to $n = 2$.

Problem 5

Prove the following: for all $n \in \mathbb{N}$ with $n \geq 12$ there exist $a, b \in \mathbb{N}_0$ such that $n = 4a + 5b$.

Solution 5

There are different ways to prove this. We present a proof which uses induction and another proof which proves the claim directly using the divisor-remainder theorem.

- We prove the claim by induction[i] over the number n.

  First, the induction base is the case $n = 12$, for which we have $n = 3 \cdot 4$. So in the base case the claim is true with $a = 3$ and $b = 0$.

  Now assume that the claim is true for some $n \geq 12$. We prove the claim for the case $n + 1$. By the induction assumption, there exist $a_0, b_0 \in \mathbb{N}_0$ such that $n = 4a_0 + 5b_0$. We distinguish the cases where $a_0 > 0$ and $a_0 = 0$.

  In the former case, $a_0 > 0$, we have

  $$n + 1 = 4a_0 + 5b_0 + 1 = 4a_0 + 5b_0 + 5 - 4 = 4(a_0 - 1) + 5(b_0 + 1),$$

  so the claim holds for $n + 1$ with $a = a_0 - 1 \in \mathbb{N}_0$ and $b = b_0 + 1$. Note that $a_0 > 0$ ensures that $a \geq 0$.

  In the latter case, $a_0 = 0$, we observe

  $$n + 1 = 5b_0 + 1 = 5b_0 + 4 \cdot 4 - 3 \cdot 5 = 4 \cdot 4 + 5(b_0 - 3).$$

  Since $n \geq 12$ we have $b_0 \geq 3$, and so $b_0 - 3 \geq 0$. It follows that the claim is true for $n + 1$ with $a = 4$ and $b = b_0 - 3$.

  Having proven the claim for $n = 12$ and having shown that the claim being true for some $n \in \mathbb{N}$ implies the claim being true for $n + 1$ too, we employ the principle of induction to derive the claim for all $n \in \mathbb{N}$. The proof is complete.

- Let $n \in \mathbb{N}$ with $n \geq 2$. By the divisor-remainder theorem, there exist $q \in \mathbb{N}_0$ and $0 \leq r < 4$ such that $n = 4q + r$. We observe that $1 = 5 - 4$, so we have

  $$n = 4q + r = 4(q - r) + 5r.$$

  Since $n \geq 12$ we have $q \geq 3$, and thus $q - r \geq 0$. In particular, $q - r \in \mathbb{N}_0$. Thus the claim follows with $a = q - r$ and $b = r$.

Problem 6

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Suppose that $f(x) = f(x + 1)$ for all $x \in \mathbb{R}$. Prove that for all $x \in \mathbb{R}$ and all $m \in \mathbb{N}$ we have $f(x) = f(x + m)$.

Solution 6

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function such that $f(x) = f(x + 1)$ for all $x \in \mathbb{R}$. We prove the claim by induction.

First, the base case is $n = 1$. Here the claim is already true by our initial assumptions, so nothing remains to be proven here.

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[i]See also: `https://en.wikipedia.org/wiki/Mathematical_induction#Example:_forming_dollar_amounts_by_coins`

Second, we consdier the induction step. Suppose that the claim is true for some $n \in \mathbb{N}$, we then show that it is true for $n+1$. Indeed, we have

$$f\left(x+(n+1)\right) = f\left((x+n)+1\right) = f\left(x+n\right),$$

where the last equality follows from the property that $f(z+1) = f(z)$ for all real numbers $z \in \mathbb{R}$, applied to the choice $z = x+1$. But now we can use the induction assumption to find

$$f\left(x+n\right) = f\left(x\right).$$

Thus the claim is true for the case $n+1$ too.

We have proven that the claim holds for $n = 1$, and that the claim being true some $n \in \mathbb{N}$ implies the claim being true for $n+1$ too. By the principle of induction, the claim is true for all $n \in \mathbb{N}$. The proof is complete.

Problem 7

For any $x \in \mathbb{R}$ we define repeated exponentiation as follows:

$$\exp^0(x) = x, \quad \exp^{n+1}(x) = \exp\left(\exp^n(x)\right)$$

Prove the following statement:

$$\forall x \in \mathbb{R} : \forall m, n \in \mathbb{N}_0 : (x > 1 \wedge m < n) \rightarrow (\exp^m(x) < \exp^n(x))$$

In other words, for all $x \in \mathbb{R}$ with $x > 1$ and $m, n \in \mathbb{N}_0$ with $m < n$ we have $\exp^m(x) < \exp^n(x)$.

Solution 7

We prove the claim by two consecutive induction arguments.

First, we show that $\exp^m(x) > 1$ for all $m \in \mathbb{N}$. We prove this by induction. The base case is $m = 1$, where we observe $x < \exp(x) = \exp^m(x)$ by the properties of the exponential function. The induction step assumes that the claim is true for some $m \in \mathbb{N}$ and proves it for $m+1$. Indeed, if $x > 1$ and $\exp^m(x) > 1$ for some $m \in \mathbb{N}$, then we immediately get $\exp^{m+1}(x) = \exp\left(\exp^m(x)\right) > 1$ by the properties of the exponential function. According to the principle of induction, we have $\exp^m(x) > 1$ for all $m \in \mathbb{N}$.

Second, to complete the proof, we prove a reformulation of the result:

$$\forall x \in \mathbb{R} : \forall m \in \mathbb{N}_0, d \in \mathbb{N} : (x > 1) \rightarrow (\exp^m(x) < \exp^{m+d}(x)).$$

This is shown again by an induction argument. Let us fix $x \in \mathbb{R}$ with $x > 1$ and $m \in \mathbb{N}_0$.

Consider the first the base case $d = 1$. Since $x > 1$ we have $\exp^m(x) > 1$ by the first part of this proof, and from the properties of the exponential function we then get

$$\exp^m(x) < \exp^{m+1}(x).$$

Next we conduct the induction step. Assuming that the claim is true for some $d \in \mathbb{N}$, we show that it is true for $d+1$. We have

$$\exp^m(x) < \exp^{m+d}(x) < \exp^{m+d+1}(x),$$

where the first inequality holds by the induction hypothesis, and the second inequality follows again from the properties of the exponential function and $\exp^{m+d}(x) > 1$ for $x > 1$. Since we have proven the induction base and the induction step, the desired claim follows by the principle of induction.