MATH 109 – HOMEWORK 6

Due Friday, February 23rd. Handwritten submissions only.
The exercises in this homework are worth 16 points.

Problem 1
Prove using the principle of induction:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

Problem 2
Let $n \in \mathbb{N}$ with $n \geq 4$. Prove that $n!$ is divisible by a square number $s \in \mathbb{N}$ with $\sqrt{s} \geq \lfloor n/2 \rfloor$.

Problem 3
Let $p \in \mathbb{N}_0$ and $n \in \mathbb{N}$. Prove Pascal's identity:

$$\sum_{l=1}^{p+1} \binom{p+1}{l} S_{n,p+1-l} = (n+1)^{p+1} - 1, \qquad \text{where} \qquad S_{n,p} := \sum_{k=1}^{n} k^p.$$

Use this formula to compute $S_{n,4}$ for arbitrary $n \in \mathbb{N}$.

Problem 4
Find the mistake in the following reasoning:

Claim: All cars have the same color.

We use the principle of induction: we show that for all $n \in \mathbb{N}$ within every set of n cars all cars have the same color.

First, if $n = 1$, then certainly every set containing 1 car has all cars with the same color.

Second, suppose the for all sets of n cars we have the cars in that set having the same color. Consider now a set $A = \{a_1, \dots, a_n, a_{n+1}\}$ of $n+1$ cars, and define

$$A_1 = \{a_2, \dots, a_n, a_{n+1}\}, \quad A_{n+1} = \{a_1, \dots, a_n\}.$$

By the induction assumption, we have that all cars in A_1 and all cars in A_{n+1} have the same color. Hence all cars in $A = A_1 \cup A_{n+1}$ have the same color.

Problem 5
Prove the following: for all $n \in \mathbb{N}$ with $n \geq 12$ there exist $a, b \in \mathbb{N}_0$ such that $n = 4a + 5b$.

Problem 6
Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Suppose that $f(x) = f(x+1)$ for all $x \in \mathbb{R}$. Prove that for all $x \in \mathbb{R}$ and all $m \in \mathbb{N}$ we have $f(x) = f(x+m)$.

Problem 7
For any $x \in \mathbb{R}$ we define repeated exponentiation as follows:

$$\exp^0(x) = x, \quad \exp^{n+1}(x) = \exp\left(\exp^n(x)\right)$$

Prove the following statement:

$$\forall x \in \mathbb{R} : \forall m, n \in \mathbb{N}_0 : (x > 1 \land m < n) \to (\exp^m(x) < \exp^n(x))$$

In other words, for all $x \in \mathbb{R}$ with $x > 1$ and $m, n \in \mathbb{N}_0$ with $m < n$ we have $\exp^m(x) < \exp^n(x)$.