MATH 109 – HOMEWORK 7

Due Friday, March 1st. Handwritten submissions only.
The exercises in this homework are worth 16 points.

Problem 1
Consider three sets X, Y, Z and two functions

$$f : X \to Y, \quad g : Y \to Z.$$

(1) Show that $g \circ f$ is injective if f and g are injective. Does the converse impliciation hold?
(2) Show that $g \circ f$ is surjective if f and g are surjective. Does the converse impliciation hold?
(3) Show that $g \circ f$ is bijective if f and g are bijective. Does the converse impliciation hold?
(4) Give an example of surjective f and injective g such that $g \circ f$ is not bijective.

Solution 1 (1) Let $x, x' \in X$ and suppose $g \circ f(x) = g \circ f(x')$, i.e. $g(f(x)) = g(f(x'))$. Then by injectivity of g, we must have $f(x) = f(x')$, which by injectivity of f implies $x = x'$. Hence $g \circ f$ is injective.
Counterexample for the converse: $f : \{0\} \to \{0, 1\} : 0 \mapsto 0$, and $g : \{0, 1\} \to \{0\} : 0 \mapsto 0, 1 \mapsto 0$.
(2) Take $z \in Z$. By surjectivity of g, there exists $y \in Y$ such that $g(y) = z$. Also, by surjectivity of f, there exists $x \in X$ such that $f(x) = y$. But then $g(f(x)) = g(y) = z$, hence $g \circ f$ is surjective.
The same counterexample as in (1) works to disprove the converse.
(3) Follows immediately from (1) and (2). Also the same counterexample works to disprove the converse.
(4) $f : \{0, 1\} \to \{0\} : 0 \mapsto 0, 1 \mapsto 0$, $g : \{0\} \to \{0, 1\} : 0 \mapsto 0$.

Problem 2
Let X and Y be sets and let $f : X \to Y$ be a function.

(1) Prove the *monomorphism property* of the injective functions: f is injective if and only if for all sets Z and functions

$$g_1 : Z \to X, \quad g_2 : Z \to X$$

such that $f \circ g_1 = f \circ g_2$ we have already $g_1 = g_2$
(2) Prove the *epimorphism property* of the surjective functions: f is surjective if and only if for all sets Z and functions

$$g_1 : Y \to Z, \quad g_2 : Y \to Z$$

such that $g_1 \circ f = g_2 \circ f$ we have already $g_1 = g_2$.

Solution 2
(1) ($\Rightarrow$). Suppose f is injective and let g_1, g_2 be any two functions from Z to X such that $f \circ g_1 = f \circ g_2$. This means that

$$\forall z \in Z : \quad f(g_1(z)) = f(g_2(z)).$$

Since f is injective, this implies

$$\forall z \in Z : \quad g_1(z) = g_2(z),$$

hence $g_1 = g_2$.
($\Leftarrow$) Suppose the monomorphism property holds for f. We want to show f is injective. So take any $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$. We choose $Z = \{z_0\}$ and define the functions $g_1, g_2 : Z \to X$ by $g_1(z_0) = x_1$ and $g_2(z_0) = x_2$. By our assumption, it follows that

$$f(g_1(z_0)) = f(x_1) = f(x_2) = f(g_2(z_0)),$$

i.e. $f \circ g_1 = f \circ g_2$. By the monomorphism property, it follows that $g_1 = g_2$, which by definition means $x_1 = x_2$. Hence f is injective.

(2) ($\Rightarrow$) Suppose f is surjective and let g_1, g_2 be any two functions from Y to Z such that $g_1 \circ f = g_2 \circ f$. Take any $y \in Y$. By surjectivity of f, there exists $x \in X$ such that $f(x) = y$. But then by assumption

$$g_1(y) = g_1(f(x)) = g_2(f(x)) = g_2(y).$$

Since $y \in Y$ was arbitrary, this means $g_1 = g_2$.
($\Leftarrow$) Suppose the epimorphism property holds for f. We want to show f is surjective. Assume by contradiction f is not surjective, i.e. there exists $y_0 \in Y$ such that for all $x \in X : f(x) \neq y_0$. Choose $Z = \{0, 1\}$ and define $g_1, g_2 : Y \to Z$ by $g_1(y) = 0$ for all $y \in Y$, $g_2(y) = 0$ for all $y \in Y$ such that $y \neq y_0$, and $g_2(y_0) = 1$. Then since $f(x) \neq y_0$ for all $x \in X$, and y_0 is the only element of Y that can be mapped to 1, it follows that $g_1(f(x)) = 0 = g_2(f(x))$ for all $x \in X$. This means $g_1 \circ f = g_2 \circ f$, and by the epimorphism property, it would follow that $g_1 = g_2$. However $g_1(y_0) = 0 \neq 1 = g_2(y_0)$. Contradiction. Hence f is surjective.

Problem 3

The *Fibonacci numbers* $f_0, f_1, f_2, \ldots$ are a sequence of numbers that are defined as follows: we set $f_0 := 0$ and $f_1 := 1$, and for $k \in \mathbb{N}$ with $k \geq 2$ we have

$$f_k := f_{k-1} + f_{k-2}.$$

- Prove the following matrix idenity: for all $n \in \mathbb{N}$ we have
$$\begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n.$$
- Prove the following identity: for all $n \in \mathbb{N}$ we have
$$(-1)^n = f_{n+1} f_{n-1} - f_n^2.$$
- Prove that for all $n \in \mathbb{N}_0$ we have $f_{2n+1} = f_n^2 + f_{n+1}^2$.

Solution 3

- We prove this by induction.
  - Base case: $n = 1$: $\begin{pmatrix} f_2 & f_1 \\ f_1 & f_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^1$. Hence the base case holds.
  - Induction hypothesis: Assume that for some $k \geq 1$, $\begin{pmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^k$.
  - Induction step: We want to prove the statement for $k + 1$. For this we can calculate:
$$\begin{aligned} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{k+1} &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^k \cdot \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} f_{k+1} + f_k & f_{k+1} \\ f_k + f_{k-1} & f_k \end{pmatrix} \\ &= \begin{pmatrix} f_{k+2} & f_{k+1} \\ f_{k+1} & f_k \end{pmatrix}. \end{aligned}$$

    Here we used the induction hypothesis to get the second equality.
- Note that
$$\det \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix} = f_{n+1} f_{n-1} - f_n^2,$$
  and
$$\det \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \left(\det \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right)^n = (-1)^n.$$
  By the first part of the problem, the equality follows.

- We can calculate

$$
\begin{aligned}
\begin{pmatrix} f_{2n+1} & f_{2n} \\ f_{2n} & f_{2n-1} \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{2n} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n} \\
&= \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix} \cdot \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix} \\
&= \begin{pmatrix} f_{n+1}^2 + f_n^2 & f_{n+1}f_n + f_n f_{n-1} \\ f_n f_{n+1} + f_{n-1} f_n & f_n^2 + f_{n-1}^2 \end{pmatrix},
\end{aligned}
$$

  where we used part 1 of the problem for the first and third equalities. Comparing the top left entries of the matrices, the required formula follows.

Problem 4

Let $n \in \mathbb{N}$ and let $A \subseteq \mathbb{R}^n$ be a set.

- We call A *star-shaped with respect to* $x_0 \in A$ if there exists $x_0 \in A$ such that for all $x \in A$ the line segment from x_0 to x is contained in A, i.e.,

$$\forall x \in A : \forall \lambda \in [0,1] : \lambda x_0 + (1-\lambda)x \in A.$$

- We call X *convex* if

$$\forall x, y \in A : \forall \lambda \in [0,1] : \lambda x + (1-\lambda)y \in A.$$

Prove the following:

(1) If A is convex then A is star-shaped with respect to some point $x_0 \in A$.
(2) There exists a star-shaped set $B \subseteq \mathbb{R}^n$ that is not convex.
(3) If $A, A' \subseteq \mathbb{R}^n$ be convex. Then $A \cap A'$ is convex.
(4) Let $M \in \mathbb{R}^{n \times n}$ be an $n \times n$ matrix. If A is convex, then the following set is convex too:

$$M(A) := \{\, y \in \mathbb{R}^n \mid \exists x \in A : Mx = y \}.$$

Solution 4

(1) Let A be a convex shape. Assume A is not star shaped with respect to any point x_0. Then for all $x_0 \in A$, there exists $y \in A$ and $\lambda \in [0,1]$ such that $\lambda x_0 + (1-\lambda)y \notin A$. But this contradicts A being a convex shape, hence A must be star shaped.

(2) Consider the set A of n coordinate axes in $\mathbb{R}^n$. I claim these form a star shaped set that is not convex. Let x_0 be the origin in $\mathbb{R}^n$, $x = (0, \dots, 0, x_k, 0, \dots 0)$ a point on an axis, and $\lambda \in [0,1]$. Then

$$\lambda x_0 + (1-\lambda)x = (0, \dots, 0, (1-\lambda)x_k, \dots, 0),$$

which lies on the same axis as x. Hence A is star-shaped. To see that A is not convex, consider $x = (x_1, 0, \dots, 0) \in A$ and $y = (0, y_2, 0, \dots, 0) \in A$. Then for $\lambda \in [0,1]$ we have

$$x\lambda + (1-\lambda)y = (\lambda x_1, (1-\lambda)y_2, 0, \dots, 0) \notin A$$

So A cannot be convex.

(3) Let $A, A' \subset \mathbb{R}^n$ be convex. Consider $x, y \in A \cap A'$ and $\lambda \in [0,1]$. Then $x, y \in A$ and $x, y \in A'$. So by convexity, $\lambda x + (1-\lambda)y \in A$ and $\lambda x + (1-\lambda)y \in A'$. Therefore $\lambda x + (1-\lambda)y \in A \cap A'$. It follows that $A \cap A'$ is convex.

(4) Let $M \in \mathbb{R}^{n \times n}$ be an $n \times n$ matrix and A a convex set. We want to show

$$M(A) := \{y \in \mathbb{R}^n | \exists x \in A : Mx = y\}$$

is also convex. Let $x, y \in M(A)$ and $\lambda \in [0,1]$. By the definition of $M(A)$, $Mx_0 = x$ and $My_0 = y$ for some $x_0, y_0 \in A$. Since A is convex, we have $\lambda x_0 + (1-\lambda)y_0 \in A$, and thus

$$M(\lambda x_0 + (1-\lambda)y_0) = \lambda(Mx_0) + (1-\lambda)(My_0) = \lambda x + (1-\lambda)y \in M(A).$$

Thus $M(A)$ is convex.

Problem 5
Prove that there is no surjective function $f : \mathbb{N} \to \mathbb{R}$.

Hint: assuming that there exists such a function f, construct a real number x that is different from $f(0), f(1), f(2), \ldots$

Solution 5
Given any function $f : \mathbb{N} \to \mathbb{R}$, we will construct an $\tilde{x} \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, $f(n) \neq \tilde{x}$. First we can enumerate the natural numbers and write the decimal expansion of the values of our function:

$$
\begin{aligned}
f(1) &= a_1.b_{11}b_{12}b_{13}b_{14}... \\
f(2) &= a_2.b_{21}b_{22}b_{23}b_{24}... \\
f(3) &= a_3.b_{31}b_{32}b_{33}b_{34}... \\
f(4) &= a_4.b_{41}b_{42}b_{43}b_{44}... \\
&\vdots
\end{aligned}
$$

Then, let $x = 0.b_{11}b_{22}b_{33}b_{44}\ldots$ and define a new map such that $\widetilde{b_{kk}} = b_{kk} + 1$ if $b_{kk} \leq 8$ and $\widetilde{b_{kk}} = 0$ if $b_{kk} = 9$. We can then construct $\widetilde{x} = 0.\widetilde{b_{11}}\widetilde{b_{22}}\widetilde{b_{33}}\widetilde{b_{44}}\ldots$. Then by construction, for all $n \in \mathbb{N}$, $\widetilde{x}$ differs from $f(n)$ at b_{nn}. Therefore for all $n \in \mathbb{N}$, $f(n) \neq \widetilde{x} \in \mathbb{R}$, so f is not surjective.