MATH 109 – HOMEWORK 7

Due Friday, March 1st. Handwritten submissions only.
The exercises in this homework are worth 16 points.

Problem 1
Consider three sets X, Y, Z and two functions

$$f : X \to Y, \quad g : Y \to Z.$$

(1) Show that $g \circ f$ is injective if f and g are injective. Does the converse impliciation hold?
(2) Show that $g \circ f$ is surjective if f and g are surjective. Does the converse impliciation hold?
(3) Show that $g \circ f$ is bijective if f and g are bijective. Does the converse impliciation hold?
(4) Give an example of surjective f and injective g such that $g \circ f$ is not bijective.

Problem 2
Let X and Y be sets and let $f : X \to Y$ be a function.

(1) Prove the *monomorphism property* of the injective functions: f is injective if and only if for all sets Z and functions

$$g_1 : Z \to X, \quad g_2 : Z \to X$$

such that $f \circ g_1 = f \circ g_2$ we have already $g_1 = g_2$
(2) Prove the *epimorphism property* of the surjective functions: f is surjective if and only if for all sets Z and functions

$$g_1 : Y \to Z, \quad g_2 : Y \to Z$$

such that $g_1 \circ f = g_2 \circ f$ we have already $g_1 = g_2$.

Problem 3
The *Fibonacci numbers* $f_0, f_1, f_2, \ldots$ are a sequence of numbers that are defined as follows: we set $f_0 := 0$ and $f_1 := 1$, and for $k \in \mathbb{N}$ with $k \geq 2$ we have

$$f_k := f_{k-1} + f_{k-2}.$$

- Prove the following matrix idenity: for all $n \in \mathbb{N}$ we have
$$\begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n.$$
- Prove the following identity: for all $n \in \mathbb{N}$ we have
$$(-1)^n = f_{n+1} f_{n-1} - f_n^2.$$
- Prove that for all $n \in \mathbb{N}_0$ we have $f_{2n+1} = f_n^2 + f_{n+1}^2$.

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Problem 4
Let $n \in \mathbb{N}$ and let $A \subseteq \mathbb{R}^n$ be a set.

- We call A *star-shaped with respect to* $x_0 \in A$ if there exists $x_0 \in A$ such that for all $x \in A$ the line segment from x_0 to x is contained in A, i.e.,
$$\forall x \in A : \forall \lambda \in [0,1] : \lambda x_0 + (1-\lambda)x \in A.$$
- We call X *convex* if
$$\forall x, y \in A : \forall \lambda \in [0,1] : \lambda x + (1-\lambda)y \in A.$$

Prove the following:

(1) If A is convex then A is star-shaped with respect to some point $x_0 \in A$.
(2) There exists a star-shaped set $B \subseteq \mathbb{R}^n$ that is not convex.
(3) If $A, A' \subseteq \mathbb{R}^n$ be convex. Then $A \cap A'$ is convex.
(4) Let $M \in \mathbb{R}^{n \times n}$ be an $n \times n$ matrix. If A is convex, then the following set is convex too:
$$M(A) := \{\, y \in \mathbb{R}^n \mid \exists x \in A : Mx = y \}\,.$$

Problem 5
Prove that there is no surjective function $f : \mathbb{N} \to \mathbb{R}$.

Hint: assuming that there exists such a function f, construct a real number x that is different from $f(0), f(1), f(2), \ldots$