MATH 109 – HOMEWORK 8

Due Friday, March 8st. Handwritten submissions only.
The exercises in this homework are worth 16 points.

Problem 1
Consider the two functions

$$f : \mathbb{R} \setminus \{1,2,3\} \to \mathbb{R}, \quad x \mapsto \frac{5x^3 - 8x^2 - 27x + 18}{(x-1)(x-2)(x-3)},$$
$$g : \mathbb{R} \setminus \{0,1,2\} \to \mathbb{R}, \quad x \mapsto \frac{5x^3 + 7x^2 - 6x}{x(x-1)(x-2)},$$

Find the largest set $A \subseteq \mathbb{R}$ such that $A \subseteq \operatorname{dom}(f)$ and $A \subseteq \operatorname{dom}(g)$ and $f_{|A} = g_{|A}$.

Solution 1
Consider the set $A = \operatorname{dom}(f) \cap \operatorname{dom}(g)$. We first show that $f_{|A} = g_{|A}$.

Let $x \in A$. Then

$$5x^3 + 7x^2 - 6x = x\left(5x^2 + 7x - 6\right), \quad 5x^3 - 8x^2 - 27x + 18 = (x-3)\left(5x^2 + 7x - 6\right).$$

So we have

$$f(x) = \frac{5x^3 - 8x^2 - 27x + 18}{(x-1)(x-2)(x-3)} = \frac{5x^2 + 7x - 6}{(x-1)(x-2)},$$
$$g(x) = \frac{5x^3 + 7x^2 - 6x}{x(x-1)(x-2)} = \frac{5x^2 + 7x - 6}{(x-1)(x-2)}$$

Hence $g(x) = f(x)$. So $f_{|A} = g_{|A}$ holds.

Also we have that A is the largest set contained in $\operatorname{dom}(f) \cap \operatorname{dom}(g)$ such that $f_{|A} = g_{|A}$. Indeed, $A = \operatorname{dom}(f) \cap \operatorname{dom}(g)$, so there cannot be a larger set with that property.

Problem 2
Let X be a set. Consider the following relation $\sim$ on the power set $\mathfrak{P}(X)$:

$$\forall A, B \in \mathfrak{P}(X) : A \sim B \leftrightarrow A \cap B \neq \emptyset$$

Is $\sim$ reflexive, symmetric, or transitive? Either prove or give a counterexample.

Solution 2
Let X be a set and let $\sim$ as in the statement of the problem.

We see that R is **not** reflexive. Indeed, for any $A \in \mathfrak{P}(X)$ we have $A \cap A = A$. In particular, $\emptyset \cap \emptyset = \emptyset$. Hence $\emptyset \sim \emptyset$ cannot be true. So X is not reflexive.

We see that R is symmetric. To see that, assume that $A, B \in \mathfrak{P}(X)$ with $A \sim B$. Then we have $A \cap B \neq \emptyset$, and hence $B \cap A \neq \emptyset$, so $B \sim A$ follows. By definition, X is symmetric.

Finally, we find that R is generally **not** transitive. For example, consider $X = \{0,1,2,3\}$ and the three sets $A = \{0,1\}$, $B = \{1,2\}$, and $C = \{2,3\}$. We then have $A \cap B \neq \emptyset$ and $B \cap C \neq \emptyset$ whereas $A \cap C = \emptyset$. In other words, $A \sim B$ and $B \sim C$ but not $A \sim C$. Hence R is generally not transitive.

Remark: One can show that R is transitive if X is empty or only has one element. If X has two elements, then R is not transitive either. Can you give an example?

Problem 3
Let R be a binary relation over a set X that is both symmetric and antisymmetric. Prove that R is a subset of the equality relation over X.

Solution 3
Assume that R is a symmetric antisymmetric bindary relation over a set X. We need to show that R is a subset of the equality relation,

$$R \subseteq \{ \ (a, b) \in X \times X \mid a = b \ \} .$$

In other words, we need to show that for all $a, b \in X$ we have $a = b$ if $a \sim b$.

To show this, suppose that $a, b \in X$ with $a \sim b$. By symmetry, $b \sim a$ follows. By antisymmetry, $a \sim b$ and $b \sim a$ imply $a = b$. This had to be shown.

Problem 4
We introduce the following relation R on the rational numbers $\mathbb{Q}$. For $x, y \in \mathbb{Q}$ we write $x \sim_R y$ if there exist natural numbers $m, n \in \mathbb{N}$ such that $x^m = y^n$. Show that R is an equivalence relation.

Solution 4
To show that R is an equivalence relation we need to verify that R is reflexive, symmetric, and transitive. We check these properties one by one.

First, to see that R is reflexive we need to show that for arbitrary $a \in \mathbb{Q}$ we have $a \sim_R a$. This just means that for all $a \in \mathbb{Q}$ there exists $m, n \in \mathbb{N}$ such that $a^m = a^n$.

Let us assume that $a \in \mathbb{Q}$. Then we can pick $m = n = 1 \in \mathbb{N}$ and find $a^m = a = a^n$. Hence $a \sim_R a$ holds.

Second, to see R is symmetric we need to show that for arbitrary $a, b \in \mathbb{Q}$ we have that $a \sim_R b$ implies $b \sim_R a$.

So let assume that $a, b \in \mathbb{Q}$ with $a \sim_R b$. This means that there exist $m, n \in \mathbb{N}$ such that $a^m = b^n$. But that obviously (after renaming variables) is equivalent to saying that there exist $m, n \in \mathbb{N}$ such that $b^m = a^n$. So $b \sim_R a$ does hold indeed.

Finally, we show that R is transitive. We need to prove that for all $a, b, c \in \mathbb{Q}$ with $a \sim_R b$ and $b \sim_R c$ we have $a \sim_R c$.

So let us assume that $a, b, c \in \mathbb{Q}$ with $a \sim_R b$ and $b \sim_R c$. This means there exist $m, n \in \mathbb{N}$ with $a^m = b^n$ and there exist $p, q \in \mathbb{N}$ with $b^p = c^q$. But in that case we have

$$a^{mp} = b^{np} \wedge b^{np} = c^{nq}.$$

In particular, $a^{mp} = c^{nq}$, which shows that $a \sim_R c$.

The proof is complete.

Problem 5
Consider the following relation on the real numbers: for all $x, y \in \mathbb{R}$ we write

$$x \sim y \qquad :\Longleftrightarrow \qquad |x| \leq |y|.$$

Show that $\sim$ is reflexive and transitive, but not antisymmetric.

Solution 5
First, we show that $\sim$ is reflexive. For any $x \in \mathbb{R}$ we have $x \sim x$ if and only if $|x| \leq |x|$. But the latter is obviously true, so $x \sim x$ holds.

Second, we show that $\sim$ is transitive. Indeed, assume that $x, y, z \in \mathbb{R}$ such that $x \sim y$ and $y \sim z$. This means that $|x| \leq |y|$ and $|y| \leq |z|$, from which conclude that $|x| \leq |z|$. But that just shows $x \sim z$. Since the x, y, z have been arbitrary, we conclude that $\sim$ is transitive.

We want to show that $\sim$ is not antisymmetric. Recall that a relation $\sim$ being antisymmetric would mean that for all $a, b \in \mathbb{R}$ we have $a \sim b$ and $b \sim a$ implies $a = b$. Hence $\sim$ not being antisymmetric means that there exist $a, b \in \mathbb{R}$ such that we have $a \sim b$ and $b \sim a$ but $a \neq b$. To show that such $a, b \in \mathbb{R}$ exist it is sufficient to give a counterexample. Such is given by choosing $a = 1$ and $b = -1$. Indeed, we observe that $|a| \leq |b|$ and $|b| \leq |a|$ holds with that choice of a and b whereas $a \neq b$. Hence $\sim$ is not antisymmetric.

Remark: We have defined the relation $\sim$ over $\mathbb{R}$ in terms of the partial $\leq$ on the $\mathbb{R}_0^+$ and the used the reflexivity and transitivity of that partial order.

Problem 6
Let X be a set. Under which conditions is the empty set $\emptyset$ a partial order over X?

Solution 6
Let X be a set and let $R = \emptyset$ be the empty relation. Assume that R is a partial order. Since R is a partial order, it is a reflexive relation, which means that

$$\forall a \in X : (a, a) \in R.$$

If $X = \emptyset$, then this statement is vacuously true.

We show by contradiction that $X \neq \emptyset$ is not possible. Indeed, suppose that $X \neq \emptyset$ and let $a \in X$. Then reflexivity of R implies $(a, a) \in R$. But this contradicts X being non-empty.

So $X = \emptyset$ must hold.

Problem 7 (Ungraded)
Consider the set $\mathbb{R}$ equipped with the canonical order $\leq$. Let ∞ and $-\infty$ be two symbols and define

$$\overline{\mathbb{R}} := \{-\infty, \infty\} \cup \mathbb{R}.$$

We define a relation R on $\overline{\mathbb{R}}$ as follows:

$$\begin{gathered}\forall x, y \in \mathbb{R} : (\quad x \sim_R y \quad :\Longleftrightarrow \quad x \leq y \quad), \\ \forall x \in \overline{\mathbb{R}} : (-\infty \leq x) \wedge (x \leq \infty), \\ -\infty \leq -\infty, \qquad -\infty \leq \infty, \qquad \infty \leq \infty.\end{gathered}$$

Show that R is a partial order over $\overline{\mathbb{R}}$.

Solution 7
We verify the three properties that define a partial order: reflexivity, antisymmetry, and transitivity.

First we show reflexivity. Let $a \in \overline{\mathbb{R}}$. If $a \in \mathbb{R}$, then $a \leq a$ is already given by the reflexivity of the partial order over $\mathbb{R}$. If $a \in \{-\infty, \infty\}$, then $a \leq a$ follows from $-\infty \leq -\infty$ or $\infty \leq \infty$. We conclude that reflexivity holds.

Second we show antisymmetry. Let $a, b \in \overline{\mathbb{R}}$ such that $a \leq b$ and $b \leq a$. We want to prove that $a = b$.

If $a, b \in \mathbb{R}$, then $a = b$ follows from the antisymmetry of the partial order over $\mathbb{R}$. If $a = -\infty$, then the only choice for b is $-\infty$, in which case $a = b = -\infty$ is true. Similarly, if $a = \infty$, then the only choice for b is ∞, in which case $a = b = \infty$ is true. For all other choices of a and b the condition $a \leq b \wedge b \leq a$ does not occur.

Third we show transitivity. Let $a, b, c \in \overline{\mathbb{R}}$ with $a \leq b$ and $b \leq c$. We want to prove that $a \leq c$.

We observe that for all $a \in \overline{\mathbb{R}}$ we have $-\infty \leq a$ and $a \leq \infty$. Moreover, we have $a = -\infty$ if $a \leq -\infty$ and we have $a = \infty$ if $\infty \leq a$.

So, if $a = -\infty$, then $a \leq c$ holds. Similarly, if $c = \infty$, then $a \leq c$ holds. If $a = \infty$, then $b = \infty$ follows, and from there we get $c = \infty$. If $c = -\infty$, then $b = -\infty$ follows, and from there we get $a = -\infty$.

It remains to consider the case that $a, c \notin \{-\infty, \infty\}$, that is, $a, c \in \mathbb{R}$. We then must have $b \in \mathbb{R}$, and so $a \leq c$ follows again by the transitivity of the partial order over $\mathbb{R}$.

The proof is complete.

Remark: The last part, transitivity, requires the most effort. Can you think of different (case distinctions) for that part of the proof.

Problem 8 (Ungraded)
Let R denote the canonical order of set $N_0^8 = \{0, 1, \ldots, 7, 8\}$. Suppose that S is another partial order over N_0^8 such that $R \subseteq S$ and $(8, 0) \in S$. Show that $S = N_0^8 \times N_0^8$.

Solution 8

We want to show that $S = N_0^8 \times N_0^8$. That means, for all $a, b \in N_0^8$ we have $a \sim_S b$. So let us assume that we have picked arbitrary $a, b \in N_0^8$.

If $a \sim_R b$ (i.e., $a \leq b$), then we have $a \sim_S b$ already because $R \subseteq S$.

If instead $\neg(a \sim_R b)$, then we have $b \leq a$. However, we have $a \leq 8$, which means $a \sim_R 8$. Since $R \subseteq S$, this implies immediately $a \sim_S 8$. Moreover, we have $0 \leq b$, which means $0 \sim_R a$. Since $R \subseteq S$, this implies immediately $0 \sim_S a$. Now recall that $8 \sim_S 0$. Since we assume that S is a partial order, we use transitivity to find

$$(a \sim_S 8) \wedge (8 \sim_S 0) \wedge (0 \sim_S b) \implies (a \sim_S b)\,.$$

This had to be shown.