# Math 109 – Winter Quarter 2018 – Midterm I

Full name: ______________________________

Student ID: ______________________________

**Instructions:**

(1) Please print your full name and your student ID.

(2) Using cheatsheets, calculators, books, or phones is **not** allowed.

(3) You have 50 minutes to complete the test.

(4) Show your work.

| Problem | Points |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| $\Sigma$ | |

**Problem 1** (10 points)
Answer the following questions by checking the right box:

(1) Is it true that for all sets $A, B, C$ we have $(A \cup B) \cup (B \cup C) = A \cup B \cup C$?

☑ Yes ☐ No

(2) Let $A, B, C, D$ be sets. What is the same as $(A \cup B) \cap (C \cup D)$? Answer should have been $(A \cap C) \cup (B \cap C) \cup (A \cap D) \cup (B \cap D)$. (We only realized this after the exam, everyone got the point.)

~~☐ $(A \cap C) \cup (A \cap B) \cup (A \cap D) \cup (B \cap D)$ ☐ $(A \cup C) \cap (A \cup B) \cap (A \cup D) \cap (B \cup D)$~~

(3) What is the negation of $P \wedge Q \wedge (R \vee S)$?

☐ $\neg P \vee \neg Q \vee \neg(R \wedge S)$ ☑ $\neg P \vee \neg Q \vee (\neg R \wedge \neg S)$ ☐ $P \wedge \neg Q \wedge \neg(R \wedge S)$

(4) Is the statement $\emptyset \subseteq \mathfrak{P}(\emptyset) \setminus \{\emptyset\}$ true?

☑ Yes ☐ No

(5) Does the set $\{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ contain the empty set as an element?

☐ Yes ☑ No

(6) Is the set $\{\{0,1\}, \{0,2\}, \{0\}, \{1\}, \{2\}, \{0,1,2\}\}$ a power set of some set $A$?

☐ Yes ☑ No

(7) Does every set $A$ contain its power set $\mathfrak{P}(A)$ as a subset?

☐ Yes ☑ No

(8) What is negation of $\forall x \in A : \exists y \in B : x \cap y = \emptyset$?

☐ $\exists y \in B : \forall x \in A : x \cap y = \emptyset$ ☐ $\forall x \in A : \exists y \in B : x \cap y \neq \emptyset$ ☑ $\exists x \in A : \forall y \in B : x \cap y \neq \emptyset$

(9) Is the proposition $((\neg P \wedge Q) \vee (\neg P \wedge \neg Q)) \wedge P$ true for some choice of truth values of $P$ and $Q$?

☐ Yes ☑ No

(10) Which of the following statements is true?

☐ $\exists t \in \mathbb{Z} : \forall s \in \mathbb{Z} : ts = 1$ ☐ $\forall x \in \mathbb{R} : \exists y \in \mathbb{Q} : y = x^2$ ☑ $\neg(\exists x \in \mathbb{Q} : \neg(x^2 \neq 5))$

**Problem 2** (6 points)
Bring the following logical propositions into conjunctive normal form:

$$(1) \qquad P_1 \quad :\Longleftrightarrow \quad \neg\,(\neg A \wedge ((\neg B \vee C) \wedge (\neg A \wedge C)) \wedge \neg B)$$
$$(2) \qquad P_2 \quad :\Longleftrightarrow \quad \neg\,((\neg W \vee X \vee \neg Z) \wedge \neg(W \vee \neg X \vee \neg Y)) \vee W$$

*Solution 2.*

$$\begin{aligned}
P_1 &\Longleftrightarrow \neg\,(\neg A \wedge ((\neg B \vee C) \wedge (\neg A \wedge C)) \wedge \neg B) \\
&\Longleftrightarrow A \vee \neg\,((\neg B \vee C) \wedge (\neg A \wedge C)) \vee B \\
&\Longleftrightarrow A \vee \neg(\neg B \vee C) \vee \neg(\neg A \wedge C) \vee B \\
&\Longleftrightarrow A \vee (B \wedge \neg C) \vee (A \vee \neg C) \vee B \\
&\Longleftrightarrow (B \wedge \neg C) \vee A \vee \neg C \vee B \\
&\Longleftrightarrow A \vee \neg C \vee B
\end{aligned}$$

$$\begin{aligned}
P_2 &\Longleftrightarrow \neg\,((\neg W \vee X \vee \neg Z) \wedge \neg(W \vee \neg X \vee \neg Y)) \vee W \\
&\Longleftrightarrow \neg(\neg W \vee X \vee \neg Z) \vee (W \vee \neg X \vee \neg Y) \vee W \\
&\Longleftrightarrow (W \wedge \neg X \wedge Z) \vee W \vee \neg X \vee \neg Y \\
&\Longleftrightarrow (W \wedge \neg X \wedge Z) \vee W \vee \neg X \vee \neg Y \\
&\Longleftrightarrow (W \vee W \vee \neg X \vee \neg Y) \wedge (\neg X \vee W \vee \neg X \vee \neg Y) \wedge (Z \vee W \vee \neg X \vee \neg Y) \\
&\Longleftrightarrow (W \vee \neg X \vee \neg Y) \wedge (Z \vee W \vee \neg X \vee \neg Y) \\
&\Longleftrightarrow W \vee \neg X \vee \neg Y
\end{aligned}$$

□

**Problem 3** (6 points)
Bring the following propositions into a form where all quantifiers are at the initial position:

(1) $Q_1 \quad :\Longleftrightarrow \quad \neg(\exists x \in A : \forall y \in B : P(x,y) \vee (\forall z \in C : W(x,z)))$
(2) $Q_2 \quad :\Longleftrightarrow \quad \neg(\exists z \in C : R(z,z)) \vee (\forall x \in A : P(x) \vee \forall y \in B(x) : H(y))$
(3) $Q_3 \quad :\Longleftrightarrow \quad \exists t \in T : \neg(\exists s \in S : J(s,t) \wedge \neg(\exists z \in C : \neg(F(s,t) \vee G(z))))$

*Solution 3.*

$$\begin{aligned}
Q_1 &\Longleftrightarrow \neg(\exists x \in A : \forall y \in B : P(x,y) \vee (\forall z \in C : W(x,z))) \\
&\Longleftrightarrow \neg(\exists x \in A : \forall y \in B : \forall z \in C : P(x,y) \vee W(x,z)) \\
&\Longleftrightarrow \forall x \in A : \exists y \in B : \exists z \in C : \neg P(x,y) \wedge \neg W(x,z)
\end{aligned}$$

$$\begin{aligned}
Q_2 &\Longleftrightarrow \neg(\exists z \in C : R(z,z)) \vee (\forall x \in A : P(x) \vee \forall y \in B(x) : H(y)) \\
&\Longleftrightarrow \forall z \in C : \neg R(z,z) \vee (\forall x \in A : P(x) \vee \forall y \in B(x) : H(y)) \\
&\Longleftrightarrow \forall z \in C : \forall x \in A : \forall y \in B(x) : \neg R(z,z) \vee P(x) \vee H(y)
\end{aligned}$$

$$\begin{aligned}
Q_3 &\Longleftrightarrow \exists t \in T : \neg(\exists s \in S : J(s,t) \wedge \neg(\exists z \in C : \neg(F(s,t) \vee G(z)))) \\
&\Longleftrightarrow \exists t \in T : \neg(\exists s \in S : J(s,t) \wedge (\forall z \in C : F(s,t) \vee G(z))) \\
&\Longleftrightarrow \exists t \in T : \neg(\exists s \in S : \forall z \in C : J(s,t) \wedge (F(s,t) \vee G(z))) \\
&\Longleftrightarrow \exists t \in T : \forall s \in S : \exists z \in C : \neg J(s,t) \vee \neg(F(s,t) \vee G(z))
\end{aligned}$$

□

**Problem 4** (8 points)

(a) Prove that for all sets $A$, $B$, and $C$ we have

$$(A \cup B) \setminus (C \cup B) = (A \setminus B) \setminus C.$$

(b) Give three sets $R$, $S$, and $T$ such that each intersection of two different of them contains exactly one element.

(c) Give four sets $W$,$X$,$Y$, and $Z$ of natural numbers such that any intersection of exactly $1 \leq k \leq 4$ different of these sets contains exactly $4 - k$ numbers.

*Solution 4.*

(a) We have

$$\begin{aligned}
x \in (A \cup B) \setminus (C \cup B) &\iff (x \in (A \cup B)) \wedge (x \notin (C \cup B)) \\
&\iff ((x \in A) \vee (x \in B)) \wedge \neg((x \in C) \vee (x \in B)) \\
&\iff ((x \in A) \vee (x \in B)) \wedge ((x \notin C) \wedge (x \notin B)) \\
&\iff ((x \in A) \wedge (x \notin C) \wedge (x \notin B)) \vee ((x \in B) \wedge (x \notin C) \wedge (x \notin B)) \\
&\iff ((x \in A) \wedge (x \notin C) \wedge (x \notin B)) \vee F \\
&\iff ((x \in A) \wedge (x \notin B)) \wedge (x \notin C) \\
&\iff (x \in A \setminus B) \wedge (x \notin C) \\
&\iff x \in (A \setminus B) \setminus C.
\end{aligned}$$

(b) We can for instance take $R = S = T = \{1\}$.

(c) Let

$$\begin{aligned}
W &= \{1, 2, 3\}, \\
X &= \{1, 2, 4\}, \\
Y &= \{1, 3, 4\}, \\
Z &= \{2, 3, 4\}.
\end{aligned}$$

□

**Problem 5** (8 points)
Answer the following three questions:

(1) Let $s$ and $t$ be positive integers with $s < t$. How many integers $k$ satisfy $s < k < t$?
(2) Let $k$ and $l$ be positive integers. How many ordered pairs $(x, y)$ of positive integers satisfy $k < x$ and $x + y = l$?
(3) How many ordered pairs $(a, b)$ of positive integers $a$ and $b$ satisfy the inequalities

$$3 < a + b < 33.$$

*Solution 5.*

(a) $t - s - 1$, since there are $t - 1$ integers $k$ satisfying $k < t$ (namely $1, 2, \dots, t - 1$), and we throw away the first $s$ of them that satisfy $k \leq s$.
(b) From the given conditions it follows that

$$0 < y = l - x < l - k.$$

Case 1: $l \leq k$. In this case there are no solutions.
Case 2: $l > k$. In this case, given any integer $y$ satisfying $0 < y < l - k$, there is a unique $x$ making the given equations true, namely $x = l - y$. Hence there are as many solutions as there are integers between 0 and $l - k$, which by (1) equals $l - k - 1$.
(c) Let $x$ be any number between 3 and 33. Putting $k = 0$ and $l = x$ in (2), this gives that the number of ordered pairs $(a, b)$ satisfying $a + b = x$ equals $x - 1$. Hence the number of ordered pairs of positive integers $(a, b)$ satisfying $3 < a + b < 33$ equals

$$\sum_{x=4}^{32}(x - 1) = 3 + 4 + 5 + \cdots + 29 + 30 + 31 = 493.$$

□

**Problem 6** (8 points)
Let $n$ be a natural number. Prove that there exists exactly one integer $m$ that is closer to $\sqrt{n}$ than any other integer.

*Solution 6.* Assume by contradiction that there is more than one integer closest to $\sqrt{n}$. Since $\sqrt{n}$ is a real number, the only way this can happen, is when $\sqrt{n} = k + \frac{1}{2}$ for some $k \in \mathbb{Z}$ (in which case $k$ and $k+1$ are equally close to $\sqrt{n}$ and they are closer than any other integer). But squaring that equation, we get

$$n = k^2 + k + \frac{1}{4}.$$

Hence

$$\frac{1}{4} = n - k^2 - k.$$

Since $k$ and $n$ are integers, it would follow that $\frac{1}{4}$ is an integer, which is a contradiction. Hence there has to be a unique integer closer to $\sqrt{n}$ than any other integer.

□