MATH 109 – PRACTICE PROBLEMS FOR MIDTERM II

Problem 1
Consider the sequence of numbers $a_1, a_2, a_3, \dots$ in $\mathbb{N}_0$ that is defined recursively as follows: we set $a_1 := 0$, and for all $k \in \mathbb{N}$ we define recursively

$$a_{k+1} := 2a_k + 1.$$

Prove that the following identity is true for all $k \in \mathbb{N}$:

$$a_k = 2^{k-1} - 1.$$

Solution 1
We prove that statement is true for all $k \in \mathbb{N}$ using the principle of induction.

For the induction base, we consider the case $k = 1$, in which case we have

$$a_1 = 0 = 1 - 1 = 2^0 - 1 = 2^{1-1} - 1.$$

This shows the claim in the case $k = 1$.

Next we perform the induction step. We assume as induction hypothesis that there exists some $k \in \mathbb{N}$ for which the claim is true. We then prove the claim for the case $k + 1$. Indeed, if the statement is already true for k, then we observe

$$a_{k+1} = 2a_k + 1 = 2\left(2^{k-1} - 1\right) + 1.$$

Here we have used the induction hypothesis. We simplify the last term further:

$$2\left(2^{k-1} - 1\right) + 1 = 2 \cdot 2^{k-1} - 2 + 1 = 2^k - 1,$$

which is just the claim for the case $k + 1$.

We have seen that the claim holds for $k = 1$ and that the claim being true for some $k \in \mathbb{N}$ implies the claim being true for $k + 1$ too. By the principle of induction, the claim is true for all $k \in \mathbb{N}$.

Problem 2
Let $a, b \in \mathbb{R}$ such that $a \in \mathbb{Q}$ and $ab \notin \mathbb{Q}$. Show that $b \notin \mathbb{Q}$.

Solution 2
This is a simple proof by contradiction: assume that $a, b \in \mathbb{R}$ such that $a \in \mathbb{Q}$ and $ab \notin \mathbb{Q}$. If $b \in \mathbb{Q}$ were true, then $ab \in \mathbb{Q}$ would be true since the product of two rational numbers is again a rational number. This is a contradiction to our assumption $ab \notin \mathbb{Q}$. Hence $b \notin \mathbb{Q}$, and the proof is complete.

Problem 3
Show that there exist no natural numbers $m, n \in \mathbb{N}$ for which $18m + 6n = 1$.

Solution 3
Since for all $m, n \in \mathbb{N}$ we have $m, n > 1$, we get that $18m + 6n > m + n > 1$. The assumption $18m + 6n = 1$ leads to a contradiction immediately.

Problem 4
Show that there exist integers $u, v \in \mathbb{Z}$ such that $25u + 3701v = 1$.

Solution 4
This is a consequence of Bezout's lemma if we can show that 25 have greatest common divisor 1. An application of the Euclidean algorithm essentially uses the identities

$$\gcd(3701, 25) = \gcd(25, 1) = \gcd(1, 0) = 1. \tag{1}$$

So the greatest common divisor of 25 and 3701 is just 1. By Bezout's lemma, the statement follows immediately.

Problem 5
Find all prime numbers $p \in \mathbb{N}$ for which $p^2 - 1$ is prime.

Solution 5
Let $p \in \mathbb{N}$ be a prime number.

In the case $p = 2$ we have $p^2 - 1 = 2^2 - 1 = 4 - 1 = 3$, which is anothe prime number. So clearly 2 is a prime number that satisfies the condition stated in the problem.

Suppose that $p \neq 2$. Then $p \geq 3$ and furthermore p is odd. As a consequence, $p^2 \geq 9$ and p^2 is odd, and thus $p^2 - 1$ is even. If $p^2 - 1$ were a prime numbern, then we would have $p = 2$ because 2 is the only even prime number. But $p^2 - 1 \geq 8 > 2$, which is a slight contradiction. Hence $p > 2$ annot satisfy the condition in the statement.

To summarize, we shown that $p = 2$ is only prime number such that $p^2 - 1$ is even.

Problem 6
Show that for all $a, b \in \mathbb{N}$ we have $a^2 - 4b - 3 \neq 0$.

Solution 6
Suppose that there exist $a, b \in \mathbb{N}$ which satisfy $a^2 - 4b - 3 = 0$. We show that this leads to a contradiction and hence the statement must hold. First, since 0 is an even number, and $-4b - 3$ is always an odd number, we conclude that a^2 must be odd. This is the case if and only if a is odd too. So there exists $c \in \mathbb{Z}$ such that $a = 2c + 1$. We then find

$$\begin{aligned} 0 = a^2 - 4b - 3 &= (2c+1)^2 - 4b - 3 \\ &= 4c^2 + 4c + 1 - 4b - 3 = 4c^2 + 4c - 4b - 2. \end{aligned}$$

This is equivalent to $2 = 4c^2 + 4c - 4b$, which in turn is equivalent to

$$1 = 2c^2 + 2c - 2b.$$

But this says that the odd number 1 is the sum of even numbers, which cannot be true. Consequently, no such a and b exist, and the proof is complete.

Problem 7 (Partially also Homework)
The *Fibonacci numbers* $f_0, f_1, f_2, \ldots$ are a sequence of numbers that are defined as follows: we set $f_0 := 0$ and $f_1 := 1$, and for $k \in \mathbb{N}$ with $k \geq 2$ we have

$$f_k := f_{k-1} + f_{k-2}.$$

- Prove the following matrix idenity: for all $n \in \mathbb{N}$ we have

$$\begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n.$$

- Prove the following identity: for all $n \in \mathbb{N}$ we have
$$(-1)^n = f_{n+1}f_{n-1} - f_n^2.$$
- Prove that for all $n \in \mathbb{N}_0$ we have $f_{2n+1} = f_n^2 + f_{n+1}^2$.
- Prove that for all $n \in \mathbb{N}_0$ we have
$$f_n = \frac{\Phi^{n+1} - (1-\Phi)^{n+1}}{\sqrt{5}},$$
where $\Phi = \frac{1+\sqrt{5}}{2}$ is the *golden ratio.*

Solution 7
Will be published together with Homework solutions.

Problem 8
Suppose we have three non-empty sets X, Y, Z and two functions
$$f : X \to Y, \quad g : Y \to Z.$$

(1) Suppose that f and g are bijective. Show that $g \circ f$ is bijective.
(2) Give examples of functions such that $f \circ g$ is bijective but neither f or g are bijective themselves.
(3) Suppose that f is not injective. Can you find some non-empty set $A \subset X$ such that $f_{|A}$ is injective?

Solution 8

(1) We can list two different proofs. One proof uses inverse functions. Since f and g are bijective, they have inverse functions f^{-1} and g^{-1}. We show that $f^{-1} \circ g^{-1}$ is the inverse of $g \circ f$. Indeed,
$$\left(f^{-1} \circ g^{-1}\right) \circ (g \circ f) = f^{-1} \circ g^{-1} \circ g \circ f = f^{-1} \circ \mathrm{Id}_Y \circ f = f^{-1} \circ f = \mathrm{Id}_X,$$
$$(g \circ f) \circ \left(f^{-1} \circ g^{-1}\right) = g \circ f \circ f^{-1} \circ g^{-1} = g \circ \mathrm{Id}_Y \circ g^{-1} = g \circ g^{-1} = \mathrm{Id}_Z\,.$$
So $g \circ f$ is invertible and hence bijective.
Alternative, we can use the results from the next practice problem. Since f and g are bijective, they are by definition both injective and surjective. As a conclusion of the next problem, $g \circ f$ is both injective and surjective. Hence $g \circ f$ is bijective.
(2) We want examples of function compositions that are bijective even though the parts of the composition are not. For example, we can set
$$f : \mathbb{Z} \to \mathbb{R}, x \mapsto x,$$
$$g : \mathbb{R} \to \mathbb{Z}, x \mapsto \lfloor x \rfloor.$$
(3) Suppose that $f : X \to Y$ is not injective. For any $x \in X$ we then consider the subset $A_x := \{x\}$. Then $f_{|A_x}$ is obviously injective since its domain has only one single element.

Problem 9 (Also Homework)
Consider three sets X, Y, Z and two functions
$$f : X \to Y, \quad g : Y \to Z.$$

(1) Show that $g \circ f$ is injective if f and g are injective. Does the converse impliciation hold?
(2) Show that $g \circ f$ is surjective if f and g are surjective. Does the converse impliciation hold?
(3) Show that $g \circ f$ is bijective if f and g are bijective. Does the converse impliciation hold?
(4) Give an example of surjective f and injective g such that $g \circ f$ is not bijective.

Problem 10

Consider the natural logarithm function $\ln : \mathbb{R}^+ \to \mathbb{R}$. With respect to that function:

(1) Find the image of $(0, 1)$
(2) Find the image of $(3, 5)$
(3) Find the preimage of $\mathbb{R}$
(4) Find the preimage of (e, e^2)
(5) Find the inverse of the logarithm function. Show that ln is injective and surjective.

Solution 9

Will be published together with Homework solutions.

Solution 10 (1) $(-\infty, 0)$
(2) $(\ln 3, \ln 5)$
(3) $\mathbb{R}^+$
(4) The inverse of $\ln : \mathbb{R}^+ \to \mathbb{R}$ is the exponential function $\exp : \mathbb{R} \to \mathbb{R}^+$. Since $\ln : \mathbb{R}^+ \to \mathbb{R}$ has an inverse, it is bijective, and hence also both injective and surjective.