MATH 109 – PRACTICE PROBLEMS FOR MIDTERM II

Problem 1
Consider the sequence of natural $a_1, a_1, a_2, a_3, \ldots$ that is defined recursively as follows: we set $a_1 := 0$, and for all $k \in \mathbb{N}$ we define recursively

$$a_{k+1} := 2a_k + 1.$$

Prove that the following identity is true for all $k \in \mathbb{N}$:

$$a_k = 2^{k-1} - 1.$$

Problem 2
Let $a, b \in \mathbb{R}$ such that $a \in \mathbb{Q}$ and $ab \notin \mathbb{Q}$. Show that $b \notin \mathbb{Q}$.

Problem 3
Show that there exist no natural numbers $m, n \in \mathbb{N}$ for which $18m + 6n = 1$.

Problem 4
Show that there exist integers $u, v \in \mathbb{Z}$ such that $25u + 3701v = 1$.

Problem 5
Find all prime numbers $p \in \mathbb{N}$ for which $p^2 - 1$ is prime.

Problem 6
Show that for all $a, b \in \mathbb{N}$ we have $a^2 - 4b - 3 \neq 0$.

Problem 7 (Partially also Homework)
The *Fibonacci numbers* $f_0, f_1, f_2, \ldots$ are a sequence of numbers that are defined as follows: we set $f_0 := 0$ and $f_1 := 1$, and for $k \in \mathbb{N}$ with $k \geq 2$ we have

$$f_k := f_{k-1} + f_{k-2}.$$

- Prove the following matrix idenity: for all $n \in \mathbb{N}$ we have
$$\begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n.$$
- Prove the following identity: for all $n \in \mathbb{N}$ we have
$$(-1)^n = f_{n+1}f_{n-1} - f_n^2.$$
- Prove that for all $n \in \mathbb{N}_0$ we have $f_{2n+1} = f_n^2 + f_{n+1}^2$.
- Prove that for all $n \in \mathbb{N}_0$ we have
$$f_n = \frac{\Phi^{n+1} - (1 - \Phi)^{n+1}}{\sqrt{5}},$$
where $\Phi = \frac{1+\sqrt{5}}{2}$ is the *golden ratio*.

Problem 8
Suppose we have three non-empty sets X, Y, Z and two functions

$$f : X \to Y, \quad g : Y \to Z.$$

(1) Suppose that f and g are bijective. Show that $g \circ f$ is bijective.
(2) Give examples of functions such that $f \circ g$ is bijective but neither f or g are bijective themselves.
(3) Suppose that f is not injective. Can you find some non-empty set $A \subset X$ such that $f_{|A}$ is injective?

Problem 9 (Also Homework)
Consider three sets X, Y, Z and two functions

$$f : X \to Y, \quad g : Y \to Z.$$

(1) Show that $g \circ f$ is injective if f and g are injective. Does the converse impliciation hold?
(2) Show that $g \circ f$ is surjective if f and g are surjective. Does the converse impliciation hold?
(3) Show that $g \circ f$ is bijective if f and g are bijective. Does the converse impliciation hold?
(4) Give an example of surjective f and injective g such that $g \circ f$ is not bijective.

Problem 10
Consider the natural logarithm function $\ln : \mathbb{R}^+ \to \mathbb{R}$. With respect to that function:
(1) Find the image of $(0, 1)$
(2) Find the image of $(3, 5)$
(3) Find the preimage of $\mathbb{R}$
(4) Find the preimage of (e, e^2)
(5) Find the inverse of the logarithm function. Show that ln is injective and surjective.