Math 109 – Winter Quarter 2018 – Midterm II

Full name: ______________________________

Student ID: ______________________________

Instructions:

(1) Please print your full name and your student ID.

(2) Using calculators, books, or phones is **not** allowed.

(3) You have 50 minutes to complete the test.

(4) Show your work.

Problem	Points
1	
2	
3	
4	
5	
6	
Σ	

Problem 1 (6 points)
Prove that for all $n \in \mathbb{N}$ we have

$$n^2 = \sum_{k=1}^{n}(2k-1).$$

Solution
We prove the statement by induction.
For the induction base, we consider $n = 1$. In that case we have

$$1^2 = 1 = \sum_{k=1}^{1}(2 \cdot 1 - 1) = 1.$$

If the claim is true for n, then we show it is true for $n+1$. Indeed, we have

$$\begin{aligned}\sum_{k=1}^{n+1}(2k-1) &= \sum_{k=1}^{n}(2k-1) + 2(n+1) - 1 \\ &= n^2 + 2(n+1) - 1 \\ &= n^2 + 2n + 2 - 1 \\ &= n^2 + 2n + 1 = (n+1)^2.\end{aligned}$$

By the principle of induction, the claim follows.

Solution
The sum on the right-hand side is the sum of the first n odd numbers: $1, 3, \ldots, 2n-1$. We let A be the sum of the first n even numbers:

$$A = \sum_{k=1}^{n} 2k = 2\sum_{k=1}^{n} k = n(n+1),$$

where we have used the well-known formula for the sum of consecutive numbers. Furthermore, we let B be the sum of the first $2n$ numbers, which is the union of the first n odd and even numbers:

$$B := \sum_{k=1}^{2n} k = \frac{2n(2n+1)}{2} = n(2n+1) = 2n^2 + n.$$

We obviously have

$$\sum_{k=1}^{n}(2k-1) = B - A = 2n^2 + n - n^2 - n = n^2.$$

This had to be shown.

Solution
Let $n \in \mathbb{N}$. We compute that

$$\sum_{k=1}^{n}(2k-1) = \sum_{k=1}^{n} 2k - \sum_{k=1}^{n} 1 = 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 = n\,(n+1) - n = n^2.$$

The proof is complete.

Problem 2 (6 points)
Let $x \in \mathbb{R}$ with $x > 0$ and $p \in \mathbb{N}$. Prove that

$$(x+1)^p \geq 1 + px.$$

Solution
We prove the claim by induction.

For the induction base we consider the case $p = 1$, where we have

$$(x+1)^p = x + 1 = 1 + px.$$

In particular, the inequality holds.

For the induction step we consider we assume that the claim is true for some $p \in \mathbb{N}$ and show that it is true for $p+1$ too. Using the induction assumption, we observe:

$$(x+1)^{p+1} = (x+1)^p(x+1) \geq (1+px)(x+1) = x + 1 + px^2 + px = (p+1)x + 1. \tag{1}$$

Hence the claim is valid for the case $p+1$ too.

By the principle of induction, the claim follows.

Solution
Using the binomial theorem, we get

$$\begin{aligned}(x+1)^p = \sum_{k=0}^{p} \binom{p}{k} x^k 1^{p-k} &= \sum_{k=0}^{p} \binom{p}{k} x^k \\ &\geq \sum_{k=0}^{1} \binom{p}{k} x^k = \binom{p}{0} x^0 + \binom{p}{1} x^1 = 1 + px.\end{aligned}$$

This proves the claim.

Solution
Let $p \in \mathbb{N}$ and define functions $L, R : \mathbb{R} \to \mathbb{R}$ by setting

$$L(x) := (1+x)^p, \quad R(x) := 1 + px$$

At $x = 0$ we have that $L(0) = 1 = R(0)$. We consider the derivatives:

$$L'(x) = p\,(1+x)^{p-1}, \quad R'(x) = p.$$

For $x > 0$ we have $L'(x) \geq R'(x)$. Using the fundamental theorem of calculus we have

$$L(x) = L(0) + \int_0^x L'(t)dt, \quad R(x) = R(0) + \int_0^x R'(t)dt.$$

Since $L(0) = R(0)$ and $L'(t) \geq R'(t)$ for all $t \geq 0$, we have $L(x) \geq R(x)$, which had to be shown.

Problem 3 (6 points)
Show that for all $a, b \in \mathbb{N}$ with $a \geq 2$ we have $a \nmid b$ or $a \nmid b+1$.

Solution
We prove the claim via contradiction. Assume that $a, b \in \mathbb{N}$ with $a \geq 2$ such that it is not true that $a \nmid b$ or $a \nmid b+1$. Then $a \mid b$ and $a \mid b+1$. Hence there exist $c, d \in \mathbb{N}$ such that $b = ac$ and $b+1 = ad$. So $ac+1 = ad$, which gives

$$1 = a(d-c).$$

We have $d > c$, since 1 is positive. But now $a(d-c) > 1$ because $a \geq 2$. This is a contradiction.

Solution
We prove the claim via contradiction. Assume that $a, b \in \mathbb{N}$ with $a \geq 2$ such that it is not true that $a \nmid b$ or $a \nmid b+1$. Then $a \mid b$ and $a \mid b+1$. But we observe

$$\frac{b+1}{a} = \frac{b}{a} + \frac{1}{a}.$$

By assumption, $\frac{b}{a}, \frac{b+1}{a} \in \mathbb{Z}$. Hence $\frac{1}{a}$ is an integer. But this can only be true if $a = 1$. Since $a \geq 2$, we get a contradiction.

Problem 4 (6 points)
Prove the following statements about the greatest common divisor:

(1) Let $a, b, d \in \mathbb{N}$. Show that if $d \mid a$ and $d \mid b$, then $d \mid \gcd(a, b)$.
Hint: Use Bezout's Lemma.

(2) Let $a, b, c \in \mathbb{N}$. Show that

$$\mathbf{gcd}\left(a, \mathbf{gcd}\left(b, c\right)\right) = \mathbf{gcd}\left(\mathbf{gcd}\left(a, b\right), c\right).$$

Hint: Use the previous part of the problem.

Solution

(1) Let $g := \gcd(a, b)$. By Bezout's lemma there exist $s, t \in \mathbb{Z}$ such that $g = sa + tb$. Let $d \in \mathbb{N}$ be a divisor of a and b. We find

$$\frac{g}{d} = s\frac{a}{d} + t\frac{b}{d}.$$

Since $d \mid a$ and $d \mid b$, we have $\frac{a}{d} \in \mathbb{Z}$ and $\frac{b}{d} \in \mathbb{Z}$. We conclude that $\frac{g}{d} \in \mathbb{Z}$.

(2) We write

$$d = \mathbf{gcd}\left(a, \mathbf{gcd}\left(b, c\right)\right), \quad d' = \mathbf{gcd}\left(\mathbf{gcd}\left(a, b\right), c\right),$$

We get $d \mid a$ and $d \mid \mathbf{gcd}\left(b, c\right)$, so $d \mid b$ and $d \mid c$ too. Using the previous part of the problem, we first get $d \mid \mathbf{gcd}\left(a, b\right)$ and then $d \mid \mathbf{gcd}\left(\mathbf{gcd}\left(a, b\right), c\right)$, i.e., $d \mid d'$.

Similarly We get $d' \mid \mathbf{gcd}\left(a, b\right)$ and $d' \mid c$, so $d' \mid a$ and $d' \mid b$ too. Using the previous part of the problem, we first get $d' \mid \mathbf{gcd}\left(b, c\right)$ and then $d' \mid \mathbf{gcd}\left(a, \mathbf{gcd}\left(b, c\right)\right)$, i.e., $d' \mid d$.

We have shown that $d' \mid d$ and $d \mid d'$. In other words, $\frac{d}{d'}$ and $\frac{d'}{d}$ are both integers. So these ratios must be 1 and hence $d = d'$. This had to be shown.

Problem 5 (6 points)
Consider the function

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto x^4 - 2.$$

With respect to this function:

- Describe the image of the interval $(-3, 3)$,
- Describe the preimage of the interval $[-5, 5]$,
- Describe the preimage of the interval $(6, \infty)$,
- Describe the preimage of the set $\{14\}$.
- Describe an open interval A such that the restriction of f to A is injective.

Solution

- $(-2, 79)$
- $\left[-\sqrt[4]{7}, \sqrt[4]{7}\right]$
- $\left(-\infty, \sqrt[4]{8}\right) \cup \left(\sqrt[4]{8}, \infty\right)$
- $\{-2, 2\}$
- For example, $(0, 1)$ or $(-3, -1)$ or $(0, \infty)$ or ...

Problem 6 (6 points)
Suppose we have three non-empty sets X, Y, Z and two functions

$$f : X \to Y, \quad g : Y \to Z.$$

(1) Write down what it means when f is injective.
(2) Write down what it means when g is surjective.
(3) Prove the following statement:
If the composition $g \circ f$ is bijective, then f is injective and g is surjective.

Solution

(1) Possible definitions include:
- f is injective if for all $x, x' \in X$ with $x \neq x'$ we have $f(x) \neq f(x')$.
- f is injective if for all $x, x' \in X$ with $f(x) = f(x')$ we have $x = x'$.
- f is injective if for all $x, x' \in X$ we have $f(x) = f(x')$ if and only if $x = x'$.
- f is injective if for all $x, x' \in X$ we have $f(x) \neq f(x')$ if and only if $x \neq x'$.

(2) Possible definitions include:
- g is surjective if for all $y \in Y$ there exists $x \in X$ such that $f(x) = y$.
- $f(X) = Y$.

(3) Let $g \circ f$ be bijective. Then $g \circ f$ is both injective and surjective.

We first show that g is surjective. Since $g \circ f$ is surjective, for every $z \in Z$ there exists $x \in X$ with $g \circ f(x) = z$. Let $y := f(x)$, so $g(y) = z$. So for every $z \in Z$ there exists $y \in Y$ with $g(y) = z$, i.e., g is surjective.

We now show that f is injective. If f were not injective, then there would exist $x, x' \in X$ such that $x \neq x'$ but $f(x) = f(x')$. But then we have $g \circ f(x) = g \circ f(x')$ even though $x = x'$, which contradicts $g \circ f$ being injective.