MATH 109 – PRACTICE FINAL

Ungraded. To be expanded.

Problem 1
Show that n^2 is odd whenever n is an odd integer.

Problem 2
Show that n^2 is even whenever n is an even integer.

Problem 3
Show that an integer n is odd whenever n^2 is odd.

Problem 4
Prove that $\sqrt[3]{2}$ is irrational.

Problem 5
Prove that $\sqrt{10}$ is irrational.

Problem 6
Prove that $\sqrt{7}$ is irrational.

Problem 7
Show that for all $x, y \in \mathbb{Z}$ we have $x^2 - 4y - 2 \neq 0$.

Problem 8
Show that for all $x, y \in \mathbb{Z}$ we have $x^2 - 4y - 3 \neq 0$.

Problem 9
Find non-negative integers $m, n \in \mathbb{N}_0$ such that $31m + 21n = 1770$.

Remark: This problem is from a 1920 mathematics textbook and alledgedly goes back to classes taught by Leonard Euler.

Problem 10
Prove that for all $a, b, m \in \mathbb{N}$ with $a \neq 0$ or $b \neq 0$ we have $\mathbf{gcd}(ma, mb) = m\,\mathbf{gcd}(a, b)$

Problem 11
Consider three natural numbers $a, b, c \in \mathbb{N}$.

(1) What does it mean for a number $g \in \mathbb{N}$ to be the greatest common divisor of a, b, and c? Describe in your own words and using formal logic.
(2) Prove that $g = \mathbf{gcd}(a, \mathbf{gcd}(b, c))$.

Problem 12
Consider the set $X := \mathbb{R}^3 \setminus \{(0, 0, 0)\}$, the set of three-dimensional vectors without the zero vector $(0, 0, 0)$.

We define relations R and S as follows:

$$R := \{(v, w) \in X \times X \mid v = -w \vee v = w\},$$
$$S := \{(v, w) \in X \times X \mid \exists \alpha \in \mathbb{R} \setminus \{0\} : v = \alpha w\}.$$

(1) Try to formulate what these relations mean. What does it mean when two vectors are related via R or S?
(2) Show that R is an equivalence relation.
(3) Show that S is an equivalence relation.
(4) Show that for all $x, y \in X$ we have $x \sim_R y$ implies $x \sim_S y$.

Problem 13

Suppose that X is a set with an equivalence relation $R \subseteq X \times X$. For any $a \in X$ we let $[a]_R$ denote its equivalence class with regards to R. Show the following:

(1) For all $a \in X$ we have $a \in [a]_R$.
(2) For all $a, b \in X$ with $b \in [a]_R$ we have $[b]_R = [a]_R$.
(3) For all $a, b \in X$ with $b \notin [a]_R$ we have $[a]_R \cap [b]_R = \emptyset$.