Part 2: Set Theory

Classes

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Outline

- Every set is a class, but some sets are not classes.
- Some classes are too large to be sets.
- Classes are an informal notion in set-theoretically axiomatized mathematics.
- Helpful notion to resolve Russell's paradox and other paradoxes of class theory.
- We discuss classes in Math 109 to:
  - understand how to resolve the paradoxes of naive set theory,
  - Sharpen our understanding of sets.

EVERYTHING IS WRONG

Let R be the set of all sets that do not contain themselves

$$R := \{\, x \mid x \notin x \,\}$$

If $R \in R$, then if $R \notin R$, and if $R \notin R$, then $R \in R$. Hence

$$R \in R \iff R \notin R.$$

This is known as Russell's paradox.

What's the mistake in the argument?

What is a class

"Class" is an informal notion in rigorous mathematics.

Classes can be considered as "syntactic sugar".

For any one-variable predicate $P(x)$, we have a class $\mathcal{C}$ that is the "collection" of all things that satisfy this predicate. We write

$$\mathcal{C} := \{ \, x \mid P(x) \, \}$$

and use the notation

$$\begin{aligned} x \in \mathcal{C} &\quad :\Longleftrightarrow \quad P(x), \\ x \notin \mathcal{C} &\quad :\Longleftrightarrow \quad \neg P(x). \end{aligned}$$

Question: how do classes relate to sets?

“Any set is a class”.

For a set A we define the predicate $P_A(x)$ via

$$P_A(x) \quad :\Longleftrightarrow \quad x \in A.$$

Then the class

$$\mathcal{C} := \{ \, x \mid P_A(x) \, \} = \{ \, x \mid x \in A \, \}$$

contains precisely the elements of A. We informally write

$$A = \mathcal{C}.$$

Question: Are the classes that are not sets?

For any set B and predicate $P(x)$ we recall the set comprehension

$$\begin{aligned} A &:= \{ x \in B \mid P(x) \} \\ &= \{ x \mid x \in B \land P(x) \} . \end{aligned}$$

This is a class that is also set.

Informally speaking, classes are defined via "set comprehensions" where the variable is not restricted to a base set.

Classes that are not sets are called **proper classes**.

Proper classes exist. They are too large to be a set.

Proper Classes: Example I

The class of all sets that do not contain themselves.

$$\mathcal{C} := \{ x \mid x \notin x \} .$$

This is a proper class.

Suppose that there exists a set C such that for all x we have $x \in C$ if and only if $x \notin x$. If $C \in \mathcal{C}$, then $C \notin \mathcal{C}$, and if $C \notin C$, then $C \in C$. Hence we get the contradiction

$$C \in C \iff C \notin C.$$

There is not set of all sets that do not contain themselves.

Technically, there is not way to construct such a set.

Proper Classes: Example II

The class of all sets

$$\mathcal{C} := \{ \, x \mid T \, \} \, .$$

This is a proper class.

Assume that C is a set that contains all sets. Then we can use set comprehension to define the set of all sets that do not contain themselves,

$$\{ \, x \in C \mid x \notin x \, \} \, ,$$

which again leads to a contradiction.

There is not set of all sets.

Technically, there is not way to construct such a set.

Review:

- Every set is a class but some classes are not sets.
- The notion of class helps us to understand the paradoxes of naive set theory and the limits in constructing classes.
- Classes are not "things". We talk about them as things in a very limited scope that caters our intuition, **but that is just syntax.**
- **Set Comprehension requires a base class to actually comprehend a set!**
- **Warning:** Outside of set theory, "class" is often used synonymously with "set" due to historical legacy.