Part 1: Formal Logic

Logical Connectives II: Laws of Formal Logic

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January 16, 2018

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Math 109

In as much as we think of logical connectives as arithmetic operations on statements, we may think of the following laws of logic as rules of computations.

Commutative Law:

$$P \wedge Q \iff Q \wedge P,$$
$$P \vee Q \iff Q \vee P.$$

Associate Law:

$$P \wedge (Q \wedge R) \iff (P \wedge Q) \wedge R,$$
$$P \vee (Q \vee R) \iff (P \vee Q) \vee R.$$

Distributive Law:

$$P \wedge (Q \vee R) \iff (P \wedge Q) \vee (P \wedge R),$$
$$P \vee (Q \wedge R) \iff (P \vee Q) \wedge (P \vee R).$$

Idempotency Law:

$$P \wedge P \iff P,$$
$$P \vee P \iff P.$$

Absorption Law:

$$P \wedge (P \vee Q) \iff P,$$
$$P \vee (P \wedge Q) \iff P.$$

Double Negation Law:

$$\neg\neg P \iff P.$$

DeMorgan's Law:

$$\neg(P \wedge Q) \iff \neg P \vee \neg Q,$$
$$\neg(P \vee Q) \iff \neg P \wedge \neg Q.$$

Example

A fun consequence of DeMorgan's Law:

$$P \wedge Q \iff \neg(\neg P \vee \neg Q),$$
$$P \vee Q \iff \neg(\neg P \wedge \neg Q),$$

In other words, negation together with either disjunction or conjunction is sufficient to express all logical operations we have encountered so far.

Example

- It is false that ((10 is prime) or (10 is a square number))

  $\iff$

  (It is false that 10 is prime) and (it is false that 10 is a square number).

- It is false that ((7 is prime) and (7 is a square number))

  $\iff$

  (It is false that 7 is prime) or (It is false that 7 is a square number)

We may use T and F in logical operations. The following holds:

$$\neg T \iff F, \qquad \neg F \iff T,$$
$$T \vee P \iff T, \qquad F \vee P \iff P,$$
$$T \wedge P \iff P, \qquad F \wedge P \iff F.$$

Suppose that two statements A and B are equivalent, i.e., $A \iff B$. Then we may replace A by B (and vice versa) in logical operations.

$$A \vee P \iff B \vee P, \quad A \wedge P \iff B \wedge P, \quad \neg A \iff \neg B.$$

The following equivalences are frequently used:

$$P \wedge \neg P \iff F, \quad P \vee \neg P \iff T.$$

Example

$$
\begin{aligned}
& (\neg P \lor Q) \land (Q \lor P) \\
\iff\ & (Q \lor \neg P) \land (Q \lor P) \\
\iff\ & Q \lor (\neg P \land P) \\
\iff\ & Q \lor F \\
\iff\ & Q.
\end{aligned}
$$

We have considered equivalences between composed statements.
What about implications?

$$P \wedge Q \implies P, \qquad P \implies P \vee Q.$$

From your computer science classes:

$$A \wedge B \wedge (C \vee \neg D)$$

As bit-operations in C, C++ or Java:

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A & B & ( C | ~ D )
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As logical operations in C, C++ or Java:

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A && B && ( C || ! D )
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As logical operation in Python:

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A and B and ( C or not D )
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Questions?