Part 4: Functions
Basic Notions

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February 23, 2018

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Math 109

What is a function?

A function $f : X \to Y$ from a set X to another set Y associates to each $x \in X$ an $f(x) \in Y$.

We call X the **domain** of the function f and we call Y the **codomain** of the function. We write

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Remark: Not only the values of the function, but also the domain and codomain are part of what distinguishes a function from other functions.

Example

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto e^x x + 6x,$$

$$f : \mathbb{R} \setminus \{2\} \to \mathbb{R}, \quad x \mapsto \frac{1}{x-2} + x^2,$$

$$g : \mathbb{R} \to \mathbb{Z}, \quad x \mapsto \lfloor x \rfloor,$$

$$t : \mathbb{N} \to \mathbb{R}, \quad n \mapsto \frac{n(n+1)}{2},$$

$$P : \mathbb{N} \to \mathfrak{P}(\mathbb{N}), \quad n \mapsto \{1, \ldots, n\},$$

$$p : \mathbb{N} \to \{0, 1\}, \quad n \mapsto \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

$$\mathrm{Id}_X : X \to X, \quad x \mapsto x$$

Example

$$\sin : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \sin(x),$$

$$\sin : (-\pi, \pi) \to \mathbb{R}, \quad x \mapsto \sin(x),$$

$$w : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \begin{cases} 0 & \text{if } x = 0, \\ \sin(1/x) & \text{otherwise.} \end{cases}$$

$$1_{\mathbb{Q}} : \mathbb{R} \to \{0, 1\}, \quad x \mapsto \begin{cases} 0 & \text{if } x \notin \mathbb{Q}, \\ 1 & \text{if } x \in \mathbb{Q}. \end{cases}$$

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4. Two functions $f : X \to Y$ and $g : X' \to Y'$ are called equal if their domain and codomain are the same, and if they agree on every point of their domain. In other words, we have $f = g$ if and only if

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5. If we are not sure about the domain or codomain, then we often state simply state that $f(x) = g(x)$ for all $x \in A \subseteq X$ in some set A that is of interest.

The range of functions

The **range** of a function $f : X \to Y$ is the set

$$\operatorname{rng} f := \{y \in Y \mid \exists x \in X : f(x) = y\}.$$

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Some authors also call this the **image** *of the function f and write the range as* $\operatorname{Im} f$*.*

Image and Pre-Image: Image

Let $f : X \to Y$ be a function.

If $A \subseteq X$ is a subset of the domain, then we write

$$
\begin{aligned}
f(A) &:= \{y \in Y \mid \exists x \in A : f(x) = y\} \\
&= \{f(x) \mid \exists x \in A\}.
\end{aligned}
$$

We call this the **image of** A under the function f. This is a subset of the codomain, i.e., $f(A) \subseteq Y$.

Image and Pre-Image: Pre-Image

Let $f : X \to Y$ be a function.

If $B \subseteq Y$ is a subset of the codomain, then we write

$$f^{-1}(B) := \{x \in X \mid f(x) \in B\}.$$

We call this the **preimage of** B under the function f. This is a subset of the domain, i.e., $f^{-1}(B) \subseteq X$.

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The notation f^{-1} does not imply anything about whether f is invertible or not. This is pure notation.

Function composition

Given two functions

$$f : X \to Y, \quad g : Y \to Z,$$

we can compose a new function

$$g \circ f : X \to Z, \quad x \mapsto g(f(x)).$$

The codomain of f must be a subset of the domain of g.

Function composition is associative:

$$h \circ g \circ f = (h \circ g) \circ f = h \circ (g \circ f)\,.$$

Function restriction

Consider a function $f : X \to Y$. For any subset $A \subseteq X$ of the domain we may consider the restriction of f to the set A:

$$f_{|A} : A \to Y, \quad x \mapsto f(x).$$

Whenever $B \subseteq A$ is in turn a subset of A, then

$$f_{|B} = \left(f_{|A}\right)_{|B}.$$

Injective Functions

A function $f : X \to Y$ is called **injective** or **one-to-one** if distinct members of X are mapped to distinct members of Y

$$\forall x, x' \in X : \left(x \neq x'\right) \to \left(f(x) \neq f(x')\right).$$

In other words, different inputs are mapped to different outputs.

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Equivalently, there are no distinct $x, x' \in X$ such that $f(x) = f(x')$:

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Equivalently, whenever $f(x) = f(x')$, then we have $x = x'$:

$$\forall x, x' \in X : (f(x) = f(x')) \to (x = x').$$

Example

- $f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto 2x + 1$: injective
- $g : (0, \infty) \to \mathbb{R}, \quad x \mapsto \ln(x)$: injective
- $p : \mathbb{R} \to \mathbb{R}, \quad x \mapsto x^2$: **not** injective
- $q : \mathbb{R}_0^+ \to \mathbb{R}, \quad x \mapsto x^2$: injective

Surjective Functions

A function $f : X \to Y$ is called **surjective** or **onto** for every $y \in Y$ there exists $x \in X$ such that $f(x) = y$.

$$\forall y \in Y : \exists x \in X : f(x) = y.$$

In other words, for every conceivable output there is at least one input.

Remark: The codomain is an intrinsic part of the function definition. Without it we wouldn't be able to tell whether a function is surjective or not.

Example

- $f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto 2x + 1$: surjective
- $g : (0, \infty) \to \mathbb{R}, \quad x \mapsto \ln(x)$: surjective
- $p : \mathbb{R} \to \mathbb{R}, \quad x \mapsto x^2$: **not** surjective
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A function $f : X \to Y$ is called **bijective** if it is both injective and surjective. In other words, the function puts the members of X and Y in “one-to-one correspondence“.

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A function $f : X \to Y$ is called **invertible** if there exists $f^{-1} : Y \to X$ such that

$$\mathrm{Id}_X = f^{-1} \circ f, \quad \mathrm{Id}_Y = f \circ f^{-1}.$$

We call f^{-1} the **inverse function** of f. The inverse function is uniquely defined:

$$x = f^{-1}(y) \iff y = f(x).$$

A function is invertible if and only if it is bijective.

Lemma

Let $f : X \to Y$ *and* $g : Y \to X$ *be invertible functions. If* $g \circ f = \mathrm{Id}_X$*, then* f *and* g *are inverses of each other.*

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Proof.

We see that

$$\begin{aligned} g^{-1} = g^{-1} \circ \mathrm{Id}_X &= g^{-1} \circ (g \circ f) \\ &= \left(g^{-1} \circ g\right) \circ f = \mathrm{Id}_Y \circ f = f. \end{aligned}$$

Hence $f = g^{-1}$. □

Questions?