Part 4: Functions
Set-Theoretical Notions

Martin Licht, Ph.D.

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UC San Diego
Department of Mathematics

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What is a function?

A function $f : X \to Y$ from a set X to another set Y associates to each $x \in X$ an $f(x) \in Y$.

Can we define functions in set-theoretical terms?

Graphs of Functions

Suppose that $f : X \to Y$ is a function. The **graph** of f is a subset of the Cartesian product $X \times Y$ and defined as follows:

$$\operatorname{graph} f := \{ (x, y) \in X \times Y \mid f(x) = y \} .$$

Suppose that we are given any subset $G \subseteq X \to Y$, under which conditions is G the graph of a function?

1. Every input value $x \in X$ should have an output value
2. The output value of every $x \in X$ should be unique.

Let X and Y be sets, and let $A \subseteq X \times Y$.

Then G is the graph of a function $f : X \to Y$ if for every $x \in X$ there exists $y \in Y$ with $(x, y) \in G$,

$$\forall x \in X : \exists y \in Y : (x, y) \in G,$$

and if this value $y \in Y$ is also unique:

$$\forall x \in X : \forall y, y' \in Y : \big((x, y) \in G \wedge (x, y') \in G\big) \to \big(y = y'\big)\,.$$

If these two conditions are met, then G is the graph of a function $f : X \to Y$ and we have for all $x \in X$ and $y \in Y$:

$$f(x) = y \iff (x, y) \in G$$

Let $G \subseteq X \times Y$ be the graph of a function. What can recover?

$$\mathrm{dom}(G) := \{\ x \in X \mid \exists y \in Y : (x, y) \in G\ \}$$

$$\mathrm{rng}(G) := \{\ y \in Y \mid \exists x \in X : (x, y) \in G\ \}$$

We can recover the domain and the range of the function from the graph G alone. What about the codomain?

¡Miércoles! *We cannot recover the codomain from the graph of the function alone.*

Set-Theoretical ends and odds

In terms of sets, we identify the function $f : X \to Y$ with the ordered pair

$$(\mathrm{codom}(f), G_f) \in \{Y\} \times (X \times Y)\,.$$

However, we are often content to identify the function f with its graph G_f for all practical purposes outside of foundational set theory.