**Part 2: Set Theory**

# First Definitions and Concepts

Martin Licht, Ph.D.

January 17, 2018

UC San Diego
Department of Mathematics

Math 109

## What is a set?

"A set is a gathering together into a whole of definite, distinct objects of our perception or of our thought - which are called elements of the set."

(Georg Cantor, Beiträge zur Begründung der transfiniten Mengenlehre)

**Informally, a set is a collection of "things".**

**A few examples:**

- $\mathbb{N} := \{1, 2, 3, \dots\}$,
- $\mathbb{N}_0 := \{0, 1, 2, 3, \dots\}$,
- $\mathbb{Z} := \{-3, -2, -1, 0, 1, 2, 3, \dots\}$,
- $\mathbb{Q}$, the rational numbers,
- $\mathbb{R}$, the real numbers,
- $\mathbb{C}$, the complex numbers,
- $\mathbb{R}^+$, the positive real numbers,
- $\mathbb{R}_0^+$, the non-negative real numbers,
- $\mathbb{R}^3$, the set of three component vectors,
- $\emptyset$, the empty set.

If a thing $x$ is an element of a set $A$, then we write

$$x \in A.$$

If a thing $x$ is not an element of a set $B$, then we write

$$x \notin B.$$

Every thing $x$ is **either** an element of any given set $C$ or it is not. Hence the following proposition is true:

$$x \in C \oplus x \notin C$$

In fact, the slightly weaker formulation with an inclusive-or holds as well:

$$x \in C \vee x \notin C$$

Note that

$$\begin{aligned} \neg(x \in C \vee x \notin C) &\iff \neg(x \in C) \wedge \neg(x \notin C) \\ &\iff x \notin C \wedge x \in C \iff x \in C \wedge x \notin C \end{aligned}$$

## A few examples

- $0 \notin \mathbb{N}$
- $\sqrt{2} \in \mathbb{R}$
- $\sqrt{2} \notin \mathbb{Q}$
- $e, \pi, \sqrt{3} \in \mathbb{R}$
- $3.\overline{14134} \notin \mathbb{Z}$
- $4 \in \{1, 2, 3, 4, 5, 6, 7\}$
- $25 \notin \{2, 4, 6, 8, \dots\}$

## Set equality

Two sets $A$ and $B$ are called **equal** if and only if

$$x \in A \iff x \in B$$

for every thing $x$. We then write

$$A = B$$

and otherwise we write

$$A \neq B.$$

The **empty set** $\emptyset$ is the set that does not contain any elements.

If $A$ is a set, then a set $S$ is called a **subset** of $A$ if all elements of $S$ are elements of $A$. In other words,

$$x \in S \implies x \in A.$$

We may also say that $A$ is a **superset** of $S$. We write

$$S \subseteq A$$

We occassionally encounter chains of subset-relations.

$$A \subseteq B \subseteq C \subseteq D.$$

**A few examples:**

- $\emptyset \subseteq A$
- $A \subseteq A$
- $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{R} \subseteq \mathbb{C}$
- $\{2, 3, 4\} \subseteq \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

## Notational Issues for the Subset Relation

Let $A$ be a set and let $S$ be a subset.
We call $S$ a **proper** subset of $A$ if $S \subseteq A$ and $S \neq A$. We then write

$$S \subsetneq A.$$

Generally, we sometimes write

$$B \subset C$$

for $B$ being a subset of $C$, but that notation leaves ambigious whether $B$ can be a proper subset or not.

## Defining sets via their elements

We may define a set $A$ by statings its elements:

$$A := \{1, 2, 3\}.$$

The order in which the elements are stated does not matter:

$$A := \{1, 2, 3\} = \{3, 2, 1\} = \{1, 3, 2\}$$

A set is an *unordered collection of things*. [1]

[1] See also the `set` container in the C++ STL...

## Sets defined via statements I

Let $P(x)$ be a statement that depends on a variable $x$. Then we define a set by

$$A := \{\ x \mid P(x)\ \}$$

This is also known as **set-builder notation** or **set comprehension**. [2]

The statement $P(x)$ is also known as the **predicate** of the set comprehension.

[2] Compare with **list comprehension** in programming languages such as Python or Haskell.

## Sets defined via statements II

We often specify a domain for the variable in the set comprehension.

$$A := \{ x \in B \mid P(x) \} = \{ x \mid x \in B \land P(x) \}$$

In naive set theory, one may regard this as syntactic sugar.

But in rigorous set theory, set comprehension with a pre-defined range for the variable is the only set comprehension that is defined.

**A few examples:**

- $A := \{\, x \in \mathbb{N} \mid x \textit{ is a square number} \,\}$
- $\mathbb{P} := \{\, x \in \mathbb{N} \mid x \textit{ is a prime number} \,\}$
- $B := \{\, x \in \mathbb{R} \mid x^2 \in \mathbb{N} \,\}$
- $C := \{\, x \in \mathbb{R} \mid 3x^2 - 5x + 6 = 1 \,\}$
- $D := \{\, x \in \mathbb{Q} \mid 3x^2 - 5x + 6 = 1 \,\}$
- $[a, b] := \{\, x \in \mathbb{R} \mid a \leq x \wedge x \leq b \,\}$
- $\mathbb{R}^+ := \{\, x \in \mathbb{R} \mid x > 0 \,\}$

## Sets of Sets

We may form sets of sets, sets of sets of sets, etc. For example:

- $\{\{0,1,2\},\{3,4,5\},\{6,7,9\}\}$
- $\{\{0,1\},\{0,2\},\{1,2\}\}$
- $\{\{0,1,2,\dots\},\mathbb{R},\{\mathbb{Z},\emptyset\},4711\}$
- $\{\emptyset\}$
- $\{\{\{\emptyset\}\}\}$

Informally, every set is a "thing" that can be an element of another set.

We may see later that "things" such as 0, 1, $\sqrt{2}$, are sets in axiomatic set theory!

Hence, technically, we only talk about sets of sets.

## Questions?