Part 5: Relations
Order relations

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Math 109

Antisymmetry

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Inspired by this, we introduce the notion of **antisymmetry**: for any binary relation R over a set X, we call R **antisymmetric** if

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Taking further inspiration from the less-than-or-equal relation on the real numbers, we introduce the notion of partial order.

Partial Orders

A binary relation R over a set X is called a **partial order** if it has the following properties:

- reflexive
- antisymmetric
- transitive

In other words,

- $\forall a \in X : a \sim_R a$
- $\forall a, b \in X : (a \sim_R b \land b \sim_R a) \to a = b$
- $\forall a, b, c \in X : (a \sim_R b \land b \sim_R c) \to a \sim_R c$

Example:

$\leq$ over the real numbers

Let R be given by

$$R := \{ (a, b) \in \mathbb{R} \times \mathbb{R} \mid a \leq b \} .$$

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Inclusion Order of Subsets

Let X be a set and R be the binary relation

$$R := \{ (A, B) \in \mathfrak{P}(X) \times \mathfrak{P}(X) \mid A \subseteq B \} .$$

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- $\forall A, B, C \in P(X) : (A \subseteq B \land B \subseteq C) \to A \subseteq C$

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Functions with Values in Partially Ordered Set

Let Y be a set and R be a partial order over Y. Let X be a set and let $F(X, Y)$ be the set of functions mapping from X to Y. We define a partial order $\mathbf{R}$ over $F(X, Y)$ by

$$\mathbf{R} := \{ (f, g) \in F(X, Y) \times F(X, Y) \mid \forall x \in X : f(x) \leq g(x) \}.$$

We write $f \leq g$ for $f \sim_{\mathbf{R}} g$.

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If we speak of partial orders, do we also have total orders?

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For two elements $a, b \in X$ in a partially ordered set X, it is possible that neither $a \leq b$ nor $b \leq a$ holds.

In other words, there are members that are incomparable. We are interested in partial orders that are **total**, i.e., all members are comparable:

$$\forall a, b \in X : (a \leq b) \vee (b \leq a) \,.$$

Total Orders: Definition

Let X be a set equipped with a relation $\leq$. Then we say that $\leq$ is a total order if the following three conditions hold:

- **Antisymmetry:** $\forall a, b \in X : (a \leq b) \land (b \leq a) \rightarrow (a = b)$
- **Transitivity:** $\forall a, b, c \in X : (a \leq b) \land (b \leq c) \rightarrow (a \leq c)$
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In particular, every total order is also a partial order.

Total Orders: Some Notation

Let X be a set equipped with a total order $\leq$. When then abbreviate

$$a < b \quad \iff \quad a \leq b \wedge a \neq b$$

for all $a, b \in X$.

Note: this defines another binary relation over X.

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Example:

Lexicographic ordering of X^n.

Suppose that X carries the total order $\leq$. Let $n \in \mathbb{N}$ and $\leq$ be a binary relation on X^n given as follows: (see blackboard)

Questions?