Part 3: Quantifiers

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Let E be a set and let $P(x)$ be a statement that depends on a variable x.

Existential Quantifier: The statement

$$\exists x \in E : P(x)$$

formalizes

There exists $x \in E$ such that $P(x)$ is true.

Universal Quantifiers: The statement

$$\forall x \in E : P(x)$$

formalizes

For all $x \in E$ we have that $P(x)$ is true.

Examples

- $\exists x \in \mathbb{R} : x^2 = 3$
- $\neg(\exists x \in \mathbb{Q} : x^2 = 2)$
- $\exists x, y, z \in [0, 1] : x^2 + y^2 = z^2$
- $\forall x \in \mathbb{Q} : x^2 \neq 3$
- $\forall x \in \mathbb{R} : x^2 \geq 0$
- $\forall x \in \mathbb{R} \setminus \{0\} : \exists y \in \mathbb{R} \setminus \{0\} : xy = 1$
- $\forall t \in \mathbb{Z} : -t \in \mathbb{Z}$
- $\exists t \in \mathbb{R} : \forall s \in \mathbb{R} : ts = 0$

Some simple observations:

$$\forall x \in A : \forall y \in B : P(x, y) \iff \forall y \in B : \forall x \in A : P(x, y),$$
$$\exists x \in A : \exists y \in B : P(x, y) \iff \exists y \in B : \exists x \in A : P(x, y).$$

However, we generally cannot switch existential and universal quantifiers:

$$\exists x \in A : \forall y \in B : P(x, y) \not\iff \forall y \in B : \exists x \in A : P(x, y),$$
$$\forall x \in A : \exists y \in B : P(x, y) \not\iff \exists y \in B : \forall x \in A : P(x, y).$$

Examples:

$$\exists p \in \mathbb{Z} : \exists q \in \mathbb{Z} : p + q = 0$$
$$\iff \exists q \in \mathbb{Z} : \exists p \in \mathbb{Z} : p + q = 0$$
$$\forall x \in \mathbb{R} \setminus \{0\} : \forall y \in \mathbb{Z} \setminus \{0\} : xy \neq 0$$
$$\iff \forall y \in \mathbb{Z} \setminus \{0\} : \forall x \in \mathbb{R} \setminus \{0\} : xy \neq 0$$

Counter examples:

$$\forall x \in \mathbb{R} \setminus \{0\} : \exists y \in \mathbb{R} : xy = 1 \not\iff \exists y \in \mathbb{R} : \forall x \in \mathbb{R} \setminus \{0\} : xy = 1$$

$$\forall x, y \in \mathbb{R} : \exists n \in \mathbb{N} : (y > x > 0) \to (nx > y)$$
$$\not\iff \exists n \in \mathbb{N} : \forall x, y \in \mathbb{R} : (y > x > 0) \to (nx > y)$$

$$\forall q \in \mathbb{Q} : \exists n, d \in \mathbb{Z} : q = \frac{n}{d}$$
$$\not\iff \exists n, d \in \mathbb{Z} : \forall q \in \mathbb{Q} : q = \frac{n}{d}$$

Restricting/Extending the quantified sets: Let $A \subseteq B$ be two sets and let $P(x)$ be a proposition that depends on one variable. Then

$$\forall x \in B : P(x) \implies \forall x \in A : P(x),$$
$$\exists x \in A : P(x) \implies \exists x \in B : P(x).$$

Examples of restricting the universal quantifier:

$$\forall x \in \mathbb{R} \setminus \{0\} : \exists y \in \mathbb{R} : xy = 1 \implies \forall x \in \mathbb{Q} \setminus \{0\} : \exists y \in \mathbb{R} : xy = 1$$

Examples of extending the existential quantifier:

$$\exists x \in \mathbb{Q} : \ln(x) = 0 \implies \exists x \in \mathbb{R} : \ln(x) = 0.$$

The opposite directions generally do not work:

$$\forall x \in \mathbb{N} : x > 0 \;\not\Longrightarrow\; \forall x \in \mathbb{Z} : x > 0,$$
$$\exists x \in \mathbb{R} : x^2 = 2 \;\not\Longrightarrow\; \exists x \in \mathbb{Q} : x^2 = 2.$$

Negation and Quantifiers: We have the following equivalences

$$\neg\,(\,\forall x \in A : P(x)\,) \iff \exists x \in A : \neg P(x),$$
$$\neg\,(\,\exists x \in A : P(x)\,) \iff \forall x \in A : \neg P(x),$$

"As the negation moves over a quantifier, the quantifier flips."

$$\forall x \in \mathbb{R} : x^2 \neq 2 \iff \neg(\exists x \in \mathbb{R} : x^2 = 2),$$
$$\exists x \in \mathbb{R} : x^2 + x = 0 \iff \neg(\forall x \in \mathbb{R} : x^2 + x \neq 0),$$

Quantifiers and Binary Logical Connectives:

We have

$$\begin{aligned} \forall x \in A : P(x) \wedge Q(x) &\iff (\forall x \in A : P(x)) \wedge (\forall x \in A : Q(x)) \\ \exists x \in A : P(x) \vee Q(x) &\iff (\exists x \in A : P(x)) \vee (\exists x \in A : Q(x)). \end{aligned}$$

However, we have

$$\begin{aligned} \forall x \in A : P(x) \vee Q(x) &\not\Longleftrightarrow (\forall x \in A : P(x)) \vee (\forall x \in A : Q(x)) \\ \exists x \in A : P(x) \wedge Q(x) &\not\Longleftrightarrow (\exists x \in A : P(x)) \wedge (\exists x \in A : Q(x)) \end{aligned}$$

Beispiele:

$$\exists x \in \mathbb{R} : x > 0 \land x < 0 \iff\!\!\!\!\!\!\!/\;\;\; (\exists x \in \mathbb{R} : x > 0) \land (\exists x \in \mathbb{R} : x < 0)$$

$$\forall y \in \mathbb{R} : y \geq 0 \lor y \leq 0 \iff\!\!\!\!\!\!\!/\;\;\; (\forall y \in \mathbb{R} : y \geq 0) \lor (\forall y \in \mathbb{R} : y \leq 0)$$

Pulling out quantifiers:

$$\forall n \in \mathbb{N} : (n \textit{ is prime }) \vee \exists p, q \in \mathbb{N} : p \neq 1 \wedge n = pq$$

$$\iff \forall n \in \mathbb{N} : (n \textit{ is prime }) \vee (\exists p, q \in \mathbb{N} : (p \neq 1 \wedge n = pq))$$

$$\iff \forall n \in \mathbb{N} : (\exists p, q \in \mathbb{N} : n \textit{ is prime }) \vee (\exists p, q \in \mathbb{N} : (p \neq 1 \wedge n = pq))$$

$$\iff \forall n \in \mathbb{N} : \exists p, q \in \mathbb{N} : (n \textit{ is prime }) \vee (p \neq 1 \wedge n = pq)$$

(Blackboard: all real numbers have an inverse or multiply to zero.)

(Blackboard: there exists a real number that is rational but does not have a rational root.)

A statement holding for all members of the empty set is (vacuously) true:

$$\forall x \in \emptyset : P(x) \iff T$$

The intuition is that there exists no counterexample; after all there is member of the empty set that could serve as a counterexample.

However, the statement that there exists a member of the empty set satisfying a given proposition is (vacuously) false:

$$\exists x \in \emptyset : P(x) \iff F$$

Questions?