Part 5: Relations
Definitions

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Relations

A **relation** between a set X and a set Y is a subset $R \subseteq X \times Y$ of their Cartesian product.

If $x \in X$ and $y \in Y$ such that $(x, y) \in R$, then we write $x \sim_R y$.

We say that x R-relates to y.

We call R a **binary relation** because it relates two sets.

Examples

- $R = \emptyset \subseteq X \times Y$.
- $R = X \times Y$.
- We have a relation on $\mathbb{R}$ given by

$$R = \{ (x, y) \in \mathbb{R}^2 \mid x^2 = y \} .$$

- We have a relation on $\mathbb{N}$ given by

$$R = \{ (m, n) \in \mathbb{N}^2 \mid \exists k \in \mathbb{N} : m + k = n \} .$$

  We have another relation on $\mathbb{N}$ given by

$$Q = \{ (m, n) \in \mathbb{N}^2 \mid \exists k \in \mathbb{N}_0 : m + k = n \} .$$

Examples

- The equality over a set gives rise to a relation:

$$R = \{ \, (a, b) \in X^2 \mid a = b \, \} \, .$$

- The inclusion order of a power set gives rise to a relation:

$$R = \{ \, (A, B) \in \mathfrak{P}(X)^2 \mid A \subseteq B \, \} \, .$$

- The natural order of the real numbers gives rise to relations:

$$R_1 = \{ \, (a, b) \in \mathbb{R}^2 \mid a < b \, \} \, .$$
$$R_2 = \{ \, (a, b) \in \mathbb{R}^2 \mid a \leq b \, \} \, .$$

Properties of Relations

Let $R \subseteq X \times X$ be a relation over a set X.

- We call R **reflexive** if for all $x \in X$ we have

$$x \sim_R x$$

- We call R **symmetric** if for all $x, y \in X$ we have

$$x \sim_R y \iff y \sim_R x.$$

- We call R **transitive** if for all $x, y, z \in X$ we have

$$x \sim_R y \wedge y \sim_R z \implies x \sim_R z.$$

Examples

- The strictly-smaller relation on the real numbers is transitive:

$$\forall x, y, z \in \mathbb{R} : x < y \land y < z \implies x < z.$$

- The smaller-or-equal relation on the real numbers is reflexive and transitive:

$$\forall x, y, z \in \mathbb{R} : x \leq y \land y \leq z \implies x \leq z.$$
$$\forall x \in \mathbb{R} : x \leq x.$$

- The division relation on the natural numbers is reflexive and transitive:

$$\forall a, b, c \in \mathbb{N} : a \mid b \land b \mid c \implies a \mid c.$$
$$\forall a \in \mathbb{N} : a \mid a.$$

Examples

- The subset relation on the power set of some set X is reflexive and transitive:

$$\forall A, B, C \in \mathfrak{P}(X) : X \subseteq Y \land Y \subseteq Z \implies X \subseteq Z.$$
$$\forall A \in \mathfrak{P}(X) : A \subseteq A.$$

- The inequality relation over any set X is symmetric:

$$\forall a, b \in X : a \neq b \iff b \neq a.$$

Equivalence Relations

A relation R over a set X is called an *equivalence relation* if it is

- reflexive
- symmetric
- transitive

For any $x \in X$ we define its equivalence class:

$$[x] = \{\, y \in X \mid x \sim_R y \,\}.$$

We write $X_{/R}$ for the set of equivalence classes:

$$X_{/R} := \{\, [x] \in \mathfrak{P}(X) \mid x \in X \,\}.$$

Example: Preimages under functions

Let $f : X \to Y$ be a function. We then define

$$R_f := \{ (x, x') \in X^2 \mid f(x) = f(x') \} .$$

This is an equivalence relation.

All equivalence relations are of that form: if R is an equivalence relation over X, then we consider the *quotient map*

$$/R : X \to X_{/R}, \quad x \mapsto [x].$$

Example: Modulo arithmetics

Let $p \in \mathbb{N}$. For all $x, y \in \mathbb{Z}$ we write

$$x \equiv_p y \qquad \text{or} \qquad x \equiv y \pmod{p}$$

if there exist $s \in \mathbb{Z}$ such that $x - y = sp$.

Theorem

This is an equivalence relation.

The equivalence classes of this equivalence relation are called **congruence classes** modulo p.

Integers as Equivalence Classes

Integers can be represented by differences of natural numbers including zero: $\forall z \in \mathbb{Z} : \exists a, b \in \mathbb{N}_0 : z = a - b$. This leads to the following equivalence relation.

The tuple $(a, b) \in \mathbb{N}_0^2$ represents the integer $a - b$.

For $(a, b), (c, d) \in \mathbb{N}_0^2$ we say that $(a, b) \sim_{\mathbb{Z}} (c, d)$ if $a + d = c + b$.

In this sense:

$$\mathbb{Z} := \mathbb{N}_0^2{}_{/\sim_{\mathbb{Z}}}$$

Rational Numbers as Equivalence Classes

Rational numbers can be represented by quotients of integers with non-zero divisor: $\forall r \in \mathbb{Q} : \exists p, q \in \mathbb{Z}, q \neq 0 : r = p/q$. This leads to the following equivalence relation.

The tuple $(p, q) \in \mathbb{Z} \times \mathbb{Z} \setminus \{0\}$ represents the rational number p/q.

For $(p, q), (s, t) \in \mathbb{Z} \times \mathbb{Z} \setminus \{0\}$ we say that $(p, q) \sim_{\mathbb{Q}} (s, t)$ if $pt = sq$.

In this sense:

$$\mathbb{Q} := (\mathbb{Z} \times \mathbb{Z} \setminus \{0\})_{/\sim_{\mathbb{Q}}}$$