Part 1: Formal Logic

Statements, Implication, Equivalence

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A statement is either true or false.

We also call 'true' or 'false' the truth values of a statement.

Given statements that are already known to be true or false, we can often deduce whether other statements are true or false, using logical deduction.

Logic, and by extension, mathematics, is about:

(1) deriving true statements from given statements via logical deduction,

(2) finding interesting statements whose truth value we want to know,

(3) assessing which deductions are possible.

Example

(1) Given the statement

$$3x + x^2 = x^2 + 4x + 4$$

and the usual statements about arithmetics, we deduce

$$x = -4$$

(2) Given the statement

$$y = 3 * 4 + 5^2 - 11$$

and the usual statements about arithmetics, and we deduce

$$y = 26$$

Example

(3) Goldbach's conjecture: every even integer z can be written as the sum of two prime numbers.

Truth value currently unknown. Goldbach's conjecture is of great interest in number theory. [1]

(4) Consider the following statement: If x is an integer *that is not a multiple of* 7, then $x - x = 0$.

Can we skip the assumption that x is not a multiple of 7? Yes.

[1] Movie: Fermat's room

Statements

In order to discuss logical deduction, we abbreviate statements by capital letters in this course:

$$P \quad :\Longleftrightarrow \quad \text{statement}$$

For example

$$\begin{aligned} A \quad &:\Longleftrightarrow \quad 1 + 1 = 2, \\ B \quad &:\Longleftrightarrow \quad \pi \text{ is irrational}, \\ C \quad &:\Longleftrightarrow \quad 2 + 2 = 5, \\ D \quad &:\Longleftrightarrow \quad \text{The Earth is a cube} \end{aligned}$$

Sometimes we have a class of statements parametrized by variables. We may write this for example as follows:

$$
\begin{aligned}
P(x) &:\Longleftrightarrow x = 5,\\
Q(x) &:\Longleftrightarrow x \text{ is a square number},\\
R(x) &:\Longleftrightarrow x \text{ is a prime number},\\
S(x) &:\Longleftrightarrow x \neq x\\
T(x) &:\Longleftrightarrow 2 \neq 3\\
U(x) &:\Longleftrightarrow 2 = 3
\end{aligned}
$$

$P(4)$ is false, $P(5)$ is true, $P(6)$ is false. $Q(9)$ is true, $Q(-5)$ is false, $Q(144)$ is true. $R(1)$, $R(4)$, $R(6)$, $R(91)$ are false and $R(2)$, $R(3)$, $R(5)$ are true. $S(x)$ is false for every x. $T(x)$ is always true, and it does not depend on x. $U(x)$ is always false, and it does not depend on x.

A statement may depend on more than just one single variable. For example:

$$\begin{aligned} P(a, b, c) \quad &:\Longleftrightarrow \quad a^2 + b^2 = c^2, \\ F(a, b, c, n) \quad &:\Longleftrightarrow \quad a^n + b^n = c^n, \\ G(x, y, z) \quad &:\Longleftrightarrow \quad x^2 = z, \end{aligned}$$

$P(3, 4, 5)$ is true, $P(1, 2, 3)$ is false. $P(x, y, z)$ means the same as $F(x, y, z, 2)$. $G(x, y, z)$ does not depend on y.

Implication and Equivalence

Given two statements A and B, it may happen that we conclude one statement from the other.

Example

1. John is in Berlin $\implies$ John is in Germany
2. $x = 3 \implies x^2 = 9$
3. $3x^2 + 12x + 15 = 3 \implies x = 2$ or $x = -2$

We write $A \implies B$ and call this an implication. *If A is true, then B is true.*

We call two statements A and B equivalent if A implies B and B implies A.

Example

1. Jane's car is Swedish $\iff$ Jane's car is a Volvo, Saab, or Scania.
2. $x = 3 \iff x^3 = 27$
3. $3x^2 + 12x + 15 = 3 \iff x = 2$ or $x = -2$

We write $A \iff B$ and call this an equivalence.

A is true if and only if B is true.

Example

$$x \text{ is a positive number} \implies x^2 \text{ is a positive number}$$

The converse is not true:

$$x^2 \text{ is a positive number} \quad \not\Rightarrow \quad x \text{ is a positive number}$$

For example, if $x = -3$ then $x^2 = 9$ is a positive number, but we cannot conclude that x was positive. The sign of x is lost information when taking the square. Hence

$$x \text{ is a positive number} \quad \not\Leftrightarrow \quad x^2 \text{ is a positive number}$$

When an implication is not an equivalence, it indicates that information has been lost.

Example

$$x \text{ is an even integer} \implies 2x \text{ is an even integer}$$

$$x \text{ is an even integer} \implies x^2 \text{ is an even integer}$$

Does the converse hold, i.e., do we have

$$x \text{ is an even integer} \iff 2x \text{ is an even integer}$$

$$x \text{ is an even integer} \iff x^2 \text{ is an even integer}$$

?

Example

Suppose that $x, y \geq 0$ are positive real numbers. Do we have equivalences:

$$x < y \iff 2x < 2y$$

$$x < y \iff \sqrt{x} < \sqrt{y}$$

$$x < y \iff x^2 < y^2$$

?

We often write chains of implications of statements:

$$\begin{aligned} & x = 3 \\ \implies & 2x = 6 \\ \implies & 2x - 6 = 0 \\ \implies & 4x^2 - 24x + 36 = 0 \end{aligned}$$

We often write chains of equivalences of statements:

$$\begin{aligned} & x^2 + 6x - 5 = 0 \\ \iff & x^2 + 6x + 9 = 14 \\ \iff & (x+3)^2 = 14 \\ \iff & x + 3 = \sqrt{14} \text{ or } x + 3 = -\sqrt{14} \\ \iff & x = -3 + \sqrt{14} \text{ or } x = -3 - \sqrt{14} \end{aligned}$$

Implications can be done wrong:

$$
\begin{aligned}
& a = b \\
\implies \quad & a + a = a + b \\
\implies \quad & 2a = a + b \\
\implies \quad & 2a - 2b = a + b - 2b \\
\implies \quad & 2(a - b) = a + b - 2b \\
\implies \quad & 2(a - b) = a - b \\
\implies \quad & 2 = 1
\end{aligned}
$$

The statements in a chain of equivalent statements are either all false or all true:

$$\begin{aligned} & 3+5=2+8 \\ \iff \quad & 3=2+3 \\ \iff \quad & 1=3 \end{aligned}$$

The latter statement is false, so all these statements are false.

$$\begin{aligned} & 10=10 \\ \iff \quad & 10=6+4 \\ \iff \quad & 10=6+2*2 \\ \iff \quad & 10=6+2(3-1) \end{aligned}$$

The statements may be parametrized by one or more variables.

$$\begin{aligned} P(x) \quad &:\Longleftrightarrow \quad x \text{ is even} \\ Q(x) \quad &:\Longleftrightarrow \quad 2x \text{ is divisible by } 4 \end{aligned}$$

$$P(x) \iff Q(x)$$

Example

We solve an quadratic equation:

$$\begin{aligned} & 3x^2 - 21x + 30 = 0 \\ \implies \quad & 3(x^2 - 7x + 10) = 0 \\ \implies \quad & x^2 - 7x + 10 = 0 \\ \implies \quad & (x-2)(x-5) = 0 \\ \implies \quad & x = 2 \text{ or } x = 5 \end{aligned}$$

Example

We solve an quadratic equation:

$$\begin{aligned} & 2x^2 + 4x - 4 = 0 \\ \implies \quad & x^2 + 2x - 2 = 0 \\ \implies \quad & x^2 + 2x + 1 = 3 \\ \implies \quad & (x+1)^2 = 3 \\ \implies \quad & x + 1 = \sqrt{3} \text{ or } x + 1 = -\sqrt{3} \\ \implies \quad & x = \sqrt{3} - 1 \text{ or } x = -\sqrt{3} - 1 \end{aligned}$$

Lines are defined by linear equations: $y = ax + b$.

Problem: Find the intersections of

$$y = x - 2, \quad y = 2x - 5$$

Solution: The two lines intersect at (x, y) if $y = x - 2$ and $y = 2x - 5$ are both true.

If (x, y) is a solution, then we can eliminate y and obtain $x - 2 = 2x - 5$. We derive

$$x - 2 = 2x - 5 \implies x + 3 = 2x \implies 3 = x \implies x = 3$$

Hence, if (x, y) is an intersection point, then $x = 3$. We conclude that $y = x - 2 = 1$ must hold.

Lines are defined by linear equations: $y = ax + b$.

Problem: Find the intersections of

$$y = 3x + 5, \quad y = 3x - 8$$

Solution: The two lines intersect at (x, y) if $y = 3x + 5$ and $y = 3x - 8$ are both true.

If (x, y) is a solution, then we can eliminate y and obtain $3x + 5 = 3x - 8$. We derive

$$3x + 5 = 3x - 8 \implies 5 = -8$$

Hence, if (x, y) is a solution, then $5 = -8$. We conclude that no point (x, y) is a solution.
This was a proof by contradiction.

Lemma

If p, q, n are positive integers with $n = pq$,

then $p < \sqrt{n}$ or $q < \sqrt{n}$.

Proof.

Assume that p, q, n are positive integers with $n = pq$ but that $p > \sqrt{n}$ and $q > \sqrt{n}$. Then we have

$$n = \sqrt{n} \cdot \sqrt{n} < p \cdot \sqrt{n} < p \cdot q < n.$$

Hence $n < n$ follows, which implies $n \neq n$. But this is clearly false, so the additional assumption $p > \sqrt{n}$ and $q > \sqrt{n}$ leads must be false. Hence, if p, q, n are positive integers with $n = pq$, then $p < \sqrt{n}$ or $q < \sqrt{n}$. □

Final Remark on Implications

Whenever we have an equivalence of statements $A \iff B$, then we also have implications

$$A \implies B, \quad B \implies A.$$

Sometimes we write equivalences as implications if only one direction is of interest to us.

Questions?