MATH 270A – NUMERICAL LINEAR ALGEBRA – HOMEWORK 2

Due Friday, October 26th. Handwritten submissions only.

Exercise 1
Let $A \in \mathbb{R}^{n\times n}$ be orthogonal and upper triangular. Prove that A is a diagonal matrix whose non-zero entries are from the set $\{1, -1\}$.

Solution 1
Let $A \in \mathbb{R}^{n\times n}$ be orthogonal and upper triangular. We show by induction that its columns are multiples of the standard unit vectors. Let $a^1, \dots, a^n$ denote the columns of A.

First, we note that a^1 is a multiple of e^1.

Next, let $1 \leq k < n$ and suppose that a^i is a multiple of e^i for each $1 \leq i \leq k$. So there exists $\alpha_i \in \mathbb{R} \setminus \{0\}$ for each $1 \leq i \leq k$ such that $\alpha_i e_i = a_i$. For each $1 \leq i \leq k$, the scalar product $a^i \cdot a^{k+1} = \alpha_i e^i \cdot a^{k+1} = 0$ vanishes, that is $e^i \cdot a^{k+1} = 0$, hence a^{k+1} has zeroes in its first k entries. Since A is upper triangular, it has zeroes in its entries $k+2, \dots, n$. This completes the induction step.

Iterating this argument leads to the desired claim: a^i is a multiple of e^i for each $1 \leq i \leq n$. Since A is an orthogonal matrix, the norms of its columns must be one, which means that the single non-zero entry in each column must have absolutely value one. This completes the proof.

Exercise 2
Calculate the Cholesky decomposition of the following matrix:

$$A = \begin{pmatrix} 4 & 2 & 4 & 4 \\ 2 & 10 & 5 & 2 \\ 4 & 5 & 9 & 6 \\ 4 & 2 & 6 & 9 \end{pmatrix}$$

Solution 2
The solution can be calculated using different formalisms. For example, if we compute the LU decomposition, then we get

$$\begin{pmatrix} 4 & 2 & 4 & 4 \\ 2 & 10 & 5 & 2 \\ 4 & 5 & 9 & 6 \\ 4 & 2 & 6 & 9 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ 1/2 & & & \\ 1 & 1/3 & 1 & \\ 1 & 0 & 1/2 & 1 \end{pmatrix} \begin{pmatrix} 4 & 2 & 4 & 4 \\ & 9 & 5 & 2 \\ & & 4 & 6 \\ & & & 4 \end{pmatrix}$$

The lower triangular matrix is the matrix L in the LDL^T decomposition of A. The diagonal of the upper triangular matrix is precise the diagonal of D in that decomposition. Taking the square root of D, we compute $G = LD^{\frac{1}{2}}$:

$$G = \begin{pmatrix} 2 & & & \\ 1 & 3 & & \\ 2 & 1 & 2 & \\ 2 & 0 & 1 & 2 \end{pmatrix}$$

This is precisely the Cholesky factor, which could have been also using the algorithm from the lecture.

Exercise 3
Use Gram-Schmidt orthogonalization to compute the QR decomposition of the following matrix:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 2 & 0 \\ 0 & -1 & -5 \end{pmatrix}$$

Solution 3
We write for the columns:

$$A = \begin{pmatrix} a^1 & a^2 & a^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 2 & 0 \\ 0 & -1 & -5 \end{pmatrix}$$

We compute the vectors in the Gram-Schmidt process.

$$q^1 = a^1$$

with $\|q^1\| = \sqrt{5}$. Next,

$$\begin{aligned} q^2 &= a^2 - \|q^1\|^{-2}(a^2 \cdot q^1)q^1, \\ &= \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} - \frac{4}{5} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \\ &= \frac{1}{5} \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix}, \end{aligned}$$

with $\|q^2\| = 2/\sqrt{5}$ and $r_{12} = -4/\sqrt{5}$. Finally,

$$\begin{aligned} q^3 &= a^3 - \|q^1\|^{-2}(a^3 \cdot q^1)q^1 - \|q^2\|^{-2}(a^3 \cdot q^2)q^2, \\ &= \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} - \|q^1\|^{-2}(a^3 \cdot q^1) \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - \|q^2\|^{-2}(a^3 \cdot q^2)\frac{1}{5} \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix}, \\ &= \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} - \|q^1\|^{-2}(a^3 \cdot q^1) \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - \|q^2\|^{-2}(a^3 \cdot q^2)\frac{1}{5} \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix}, = \frac{1}{5} \begin{pmatrix} -32 \\ 16 \\ 25 \end{pmatrix}, \end{aligned}$$

where we get $\|q^3\| = 29/5$ together with $r_{13} = -2/\sqrt{5}$ and $r_{23} = 4/\sqrt{5}$. Hence the QR decomposition turns out to be

$$A = \begin{pmatrix} 1 & -\frac{4}{5} & -\frac{32}{5} \\ 2 & \frac{2}{5} & \frac{16}{5} \\ 0 & 0 & \frac{25}{5} \end{pmatrix} \begin{pmatrix} \sqrt{5} & -4/\sqrt{5} & -2/\sqrt{5} \\ & 2/\sqrt{5} & 4/\sqrt{5} \\ & & 29/5 \end{pmatrix}$$

Exercise 4
Use Givens rotations to compute the QR decomposition of the following matrix:

$$A = \begin{pmatrix} 0 & 1 & -3 \\ 0 & -1 & -1 \\ 6 & 3 & 9 \end{pmatrix}$$

Solution 4
The first Givens rotation is the identity matrix, the second Givens rotation by minus 90 degree, and the

third Givens rotation is by minus 135 degree.

$$\begin{pmatrix}1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1 & -3 \\ 0 & -1 & -1 \\ 6 & 3 & 9\end{pmatrix} = \begin{pmatrix}6 & 3 & 9 \\ 0 & \sqrt{2} & -\sqrt{2} \\ 6 & 0 & -2\sqrt{2}\end{pmatrix}. \tag{1}$$

Exercise 5
Use Householder transformations to compute the upper triangular matrix in the QR decomposition of the following matrix:

$$A = \begin{pmatrix}1 & -1 & -1 & -6 \\ 1 & 1 & -1 & -2 \\ 1 & 0 & -1 & 8 \\ 1 & 0 & 1 & 8\end{pmatrix}$$

Solution 5

Exercise 6
Let A be an invertible $n \times n$ matrix, let Q be an orthogonal matrix, and let R be an upper triangular matrix such that $A = QR$.

- Show that $|\det(A)| = |\det(R)|$.
- Let $1 \leq j \leq n$. Prove that the ℓ^2-norm of the j-th column of A equals the ℓ^2-norm of the j-th column of R.
- Prove Hadamard's inequality for the determinant:

$$|\det(A)| \leq \prod_{j=1}^{n} \left(\sum_{k=1}^{n} |a_{kj}|^2 \right)^{1/2}.$$

Solution 6

- By product formula of the determinant,

$$\det(A) = \det(Q)\det(R).$$

  Since $\det(Q)$ has unit absolute value, we get $|\det(A)| = |\det(R)|$.
- Let $q^1, \dots, q^n \in \mathbb{R}^n$ be the columns of Q and let $a^1, \dots, a^n \in \mathbb{R}^n$ be the columns of A. We have

$$a^j = \sum_{k=1}^{j} r_{kj} q^k.$$

  By the orthogonality of the columns of Q we find

$$\|a^j\|^2 = \sum_{k=1}^{j} r_{kj}^2.$$

  This shows the desired claim.
- To prove Hadamard's inequality, we first use that

$$|\det(A)| = |\det(R)| = \prod_{j=1}^{n} |r_{jj}|.$$

  For each $1 \leq j \leq n$ we then get

$$|r_{jj}|^2 \leq \sum_{k=1}^{j} |r_{kj}|^2 = \|a^j\|^2 = \sum_{k=1}^{jn} |a_{kj}|^2$$

from the previous part of the problem. In combination, the desired claim follows.

Exercise 7
Prove that a symmetric matrix $A \in \mathbb{R}^{n\times n}$ is positive definite if and only if we have $\det(A_k) > 0$ for all $1 \leq k \leq n$, where $A_k \in \mathbb{R}^{k\times k}$ denotes the submatrix of A in the first k rows and columns:

$$A_k = \begin{pmatrix} a_{11} & \dots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \dots & a_{kk} \end{pmatrix}.$$

Solution 7
The claim is true for $n = 1$. We finish the proof by induction and assume that the claim of equivalence already holds for some number n.

If A is positive definite, then its first diagonal entry is positive definite. So we can perform one step of the LDL^t decomposition to obtain a matrix

$$B = \begin{pmatrix} b_{11} & 0^t \\ 0 & B' \end{pmatrix}$$

with $b_{11} > 0$ and a submatrix

$$B' = \begin{pmatrix} b_{22} & \dots & b_{2n} \\ \vdots & \ddots & \vdots \\ b_{n2} & \dots & b_{nn} \end{pmatrix}.$$

By construction, B is positive definite, and so is B'. By induction assumption, it follows that $\det(B') > 0$. Thus we get $\det(B) = b_{11}\det(B') > 0$. By the construction of the first step in the LDL^t decomposition we have $\det(A) = \det(B) > 0$.

Exercise 8
A lower triangular matrix L of size $n \times n$ can be stored in computer memory only in terms of its diagonal and subdiagonal entries.

$$L = \begin{pmatrix} l_{11} & & & \\ l_{21} & l_{22} & & \\ \vdots & \vdots & \ddots & \\ l_{n1} & l_{n2} & \dots & l_{nn} \end{pmatrix}.$$

Describe an in-place algorithm that computes the matrix-matrix product LL^T and stores the diagonal and subdiagonal entries of the result in the memory of the input.

Note that matrix LL^T is symmetric, so its superdiagonal entries are completely described by its subdiagonal entries.

Solution 8
The algorithm can compute LL^T row by row, from the bottom to the top row, and within each row, from the right-most to the left-most entry. Let's first consider an algorithm that computes the diagonal and subdiagonal parts of $A := LL^T$ and saves the result in a different memory place:

- FOR $i = n, \dots, 1$ DO
-   FOR $j = i, \dots, 1$ DO
-     $F := 0$
-     FOR $k = 1, \dots, j$ DO $F = F + L_{ik}L_{jk}$ END FOR
-     $A_{ij} = F$
-   END FOR

- END FOR

We can savely overwrite L_{ij} at the end of the second loop by the newly computed value of A_{ij} because L_{ij} is not involved in the computation of the entries in rows above the i-th row, and it is not involved in computing entries with lower column index. This leads to the final algorithm:

- FOR $i = n, \dots, 1$ DO
-    FOR $j = i, \dots, 1$ DO
-       $F := 0$
-       FOR $k = 1, \dots, j$ DO $F = F + L_{ik}L_{jk}$ END FOR
-       $L_{ij} = F$
-    END FOR
- END FOR