Exercise 1
Let \( A \in \mathbb{R}^{n \times n} \) be orthogonal and upper triangular. Prove that \( A \) is a diagonal matrix whose non-zero entries are from the set \( \{1, -1\} \).

Exercise 2
Calculate the Cholesky decomposition of the following matrix:
\[
A = \begin{pmatrix}
4 & 2 & 4 & 4 \\
2 & 10 & 5 & 2 \\
4 & 5 & 9 & 6 \\
4 & 2 & 6 & 9
\end{pmatrix}
\]

Exercise 3
Use Gram-Schmidt orthogonalization to compute the QR decomposition of the following matrix:
\[
A = \begin{pmatrix}
1 & 0 & 2 \\
2 & 2 & 0 \\
0 & -1 & -5
\end{pmatrix}
\]

Exercise 4
Use Givens rotations to compute the QR decomposition of the following matrix:
\[
A = \begin{pmatrix}
3 & 15 & 12 \\
-6 & 3 & 9 \\
6 & 0 & -6
\end{pmatrix}
\]

Exercise 5
Use Householder transformations to compute the QR decomposition of the following matrix:
\[
A = \begin{pmatrix}
1 & 2 & 3 & 0 \\
1 & 5 & 6 & 0 \\
1 & 8 & 9 & 0 \\
1 & 11 & 12 & 1
\end{pmatrix}
\]

Exercise 6
Let \( A \) be an invertible \( n \times n \) matrix, let \( Q \) be an orthogonal matrix, and let \( R \) be an upper triangular matrix such that \( A = QR \).

- Show that \(|\det(A)| = |\det(R)|\).
- Let \( 1 \leq j \leq n \). Prove that the \( \ell^2 \)-norm of the \( j \)-th column of \( A \) equals the \( \ell^2 \)-norm of the \( j \)-th column of \( R \).
• Prove Hadamard’s inequality for the determinant:

$$|\det(A)| \leq \prod_{j=1}^{n} \left( \sum_{k=1}^{n} |a_{kj}| \right)^{1/2}.$$ 

**Exercise 7**
Prove that a symmetric matrix $A \in \mathbb{R}^{n \times n}$ is positive definite if and only if we have $\det(A_k) > 0$ for all $1 \leq k \leq n$, where $A_k \in \mathbb{R}^{k \times k}$ denotes the submatrix of $A$ in the first $k$ rows and columns:

$$A_k = \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{pmatrix}.$$ 

**Exercise 8**
A lower triangular matrix $L$ of size $n \times n$ can be stored in computer memory only in terms of its diagonal and subdiagonal entries.

$$L = \begin{pmatrix} l_{11} & l_{21} & l_{22} \\ l_{21} & l_{22} & \ddots \\ \vdots & \ddots & \ddots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{pmatrix}.$$ 

Describe an in-place algorithm that computes the matrix-matrix product $LL^T$ and stores the diagonal and subdiagonal entries of the result in the memory of the input.

Note that matrix $LL^T$ is symmetric, so its superdiagonal entries are completely described by its subdiagonal entries.