MATH 270A – NUMERICAL LINEAR ALGEBRA – HOMEWORK 2

Due Friday, October 26th. Handwritten submissions only.

Exercise 1
Let $A \in \mathbb{R}^{n \times n}$ be orthogonal and upper triangular. Prove that A is a diagonal matrix whose non-zero entries are from the set $\{1, -1\}$.

Exercise 2
Calculate the Cholesky decomposition of the following matrix:

$$A = \begin{pmatrix} 4 & 2 & 4 & 4 \\ 2 & 10 & 5 & 2 \\ 4 & 5 & 9 & 6 \\ 4 & 2 & 6 & 9 \end{pmatrix}$$

Exercise 3
Use Gram-Schmidt orthogonalization to compute the QR decomposition of the following matrix:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 2 & 0 \\ 0 & -1 & -5 \end{pmatrix}$$

Exercise 4
Use Givens rotations to compute the QR decomposition of the following matrix:

$$A = \begin{pmatrix} 0 & 1 & -3 \\ 0 & -1 & -1 \\ 6 & 3 & 9 \end{pmatrix}$$

Exercise 5
Use Householder transformations to compute the upper triangular matrix in the QR decomposition of the following matrix:

$$A = \begin{pmatrix} 1 & -1 & -1 & -6 \\ 1 & 1 & -1 & -2 \\ 1 & 0 & -1 & 8 \\ 1 & 0 & 1 & 8 \end{pmatrix}$$

Exercise 6
Let A be an invertible $n \times n$ matrix, let Q be an orthogonal matrix, and let R be an upper triangular matrix such that $A = QR$.

- Show that $|\det(A)| = |\det(R)|$.
- Let $1 \leq j \leq n$. Prove that the ℓ^2-norm of the j-th column of A equals the ℓ^2-norm of the j-th column of R.

- Prove Hadamard's inequality for the determinant:

$$|\det(A)| \leq \prod_{j=1}^{n} \left(\sum_{k=1}^{n} |a_{kj}|^2 \right)^{1/2}.$$

Exercise 7

Prove that a symmetric matrix $A \in \mathbb{R}^{n\times n}$ is positive definite if and only if we have $\det(A_k) > 0$ for all $1 \leq k \leq n$, where $A_k \in \mathbb{R}^{k\times k}$ denotes the submatrix of A in the first k rows and columns:

$$A_k = \begin{pmatrix} a_{11} & \dots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \dots & a_{kk} \end{pmatrix}.$$

Exercise 8

A lower triangular matrix L of size $n \times n$ can be stored in computer memory only in terms of its diagonal and subdiagonal entries.

$$L = \begin{pmatrix} l_{11} & & & \\ l_{21} & l_{22} & & \\ \vdots & \vdots & \ddots & \\ l_{n1} & l_{n2} & \dots & l_{nn} \end{pmatrix}.$$

Describe an in-place algorithm that computes the matrix-matrix product LL^T and stores the diagonal and subdiagonal entries of the result in the memory of the input.

Note that matrix LL^T is symmetric, so its superdiagonal entries are completely described by its subdiagonal entries.