Exercise 1
Given a vector \( v \in \mathbb{R}^2 \), we want to define a rotation matrix \( G \in \mathbb{R}^{2 \times 2} \) and \( \gamma \in \mathbb{R} \) such that \( Gv = \gamma e^1 \). Write
\[
G = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}.
\]
Show that the following two constructions provide the coefficients \( c \) and \( s \) and the value \( \gamma \).

- **Variant 1:**
  \[
  \gamma = v_2 \cdot \nu, \quad s = \frac{1}{\nu}, \quad c := s \tau, \quad \tau = \frac{v_1}{v_2}, \quad \nu = \sqrt{1 + \tau^2}.
  \]

- **Variant 2:**
  \[
  \gamma = v_1 \cdot \nu, \quad c = \frac{1}{\nu}, \quad s := c \tau, \quad \tau = \frac{v_2}{v_1}, \quad \nu = \sqrt{1 + \tau^2}.
  \]

**Remark 1**
In applications, the first variant is preferred if \( |v_2| \geq |v_1| \) and the second variant is preferred if \( |v_1| \geq |v_2| \).

Exercise 2
Let \( A \in \mathbb{R}^{n \times m} \) with \( \ker(A) = 0 \) and let \( b \in \mathbb{R}^n \). Show that the matrix
\[
M = \begin{pmatrix} I & A \\ A^T & 0 \end{pmatrix}
\]
is invertible. Prove that the solution of the system
\[
\begin{pmatrix} I \\ A^T \\ 0 \end{pmatrix} \begin{pmatrix} r \\ x \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}
\]
satisfies \( A^T Ax = A^T b \).

Exercise 3
Let \( A \in \mathbb{R}^{n \times m} \) with \( \text{ran}(A) = \mathbb{R}^n \) and let \( b \in \mathbb{R}^n \). Show that the matrix
\[
M = \begin{pmatrix} -I & A^T \\ A & 0 \end{pmatrix}
\]
is invertible. Prove that the solution of the system
\[
\begin{pmatrix} -I & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}
\]
satisfies \( Ax = b \) with \( x = A^T y \).

Exercise 4
Let \( D \in \mathbb{R}^{n \times m} \) and \( T \in \mathbb{R}^{m \times n} \) be matrices whose entries are zero except for the first \( d \) diagonal entries \( D_{11}, D_{22}, \ldots, D_{dd} \) and \( T_{11}, T_{22}, \ldots, T_{dd} \), respectively, where \( d \leq \min(n, m) \). Assume that \( T_{ii} = D_{ii}^{-1} \) for \( 1 \leq i \leq d \). Show that \( T = D^f \).
Exercise 5
Let \( A \in \mathbb{R}^{n \times m} \) and \( B \in \mathbb{R}^{m \times n} \). Show that \( B = A^\dagger \) if and only if the following four conditions hold:
\[
ABA = A, \quad BAB = B, \quad (AB)^T = AB, \quad (BA)^T = BA.
\]

Proof. Let \( A = WDV^T \) be the (full) singular value decomposition of \( A \). From the lecture and the previous exercise, we know that
\[
A^\dagger = VD^\dagger W.
\]
We verify that the pseudoinverse \( A^\dagger \) satisfies the four required identities by simply plugging in the pseudoinverse and doing elementary computations.

We show that the pseudoinverse \( A^\dagger \) is the only solution to the four equations
\[
ABA = A, \quad BAB = B, \quad (AB)^T = AB, \quad (BA)^T = BA.
\]
One easily checks that \( (A^T)^\dagger = (A^\dagger)^T \) using the singular value decomposition. Now we find
\[
A^\dagger AB = A^\dagger ABAB = A^\dagger AA^\dagger B^T B = A^T (A^\dagger)^T A^T B^T B = A^T B^T B = (BA)^T B = BAB = B
\]
Taking the transpose of the first two conditions, we get
\[
A^T B^T A^T = A^T \quad \text{and} \quad B^T A^T B^T = B^T.
\]
Now we find
\[
A^\dagger AB = A^\dagger B^T A^T = A^\dagger AA^\dagger B^T A^T = A^\dagger A^\dagger A^\dagger B^T A^T = A^\dagger A^\dagger B\quad \text{and} \quad A^\dagger = A^\dagger.
\]
This completes the proof. \( \square \)

Exercise 6
Let \( T : \mathbb{R}^n \to \mathbb{R}^n \) be a contraction with contraction constant \( q \in [0, 1) \). Let \( x^* \in \mathbb{R}^n \) be the unique fixpoint of \( T \). Let \( x^{(0)} \in \mathbb{R}^n \) and define recursive \( x^{(m+1)} = T(x^{(m)}) \) for \( m \in \mathbb{N}_0 \).

Let \( \epsilon > 0 \). Find the minimum number \( M \) for which \( \|x^{(M)} - x^*\| < \epsilon \) in terms of \( \epsilon \) and
(a) the initial error norm \( \|x^0 - x^*\| \).
(b) the initial difference \( \|x^0 - x^1\| \).

Exercise 7
Consider the fixpoint iteration
\[
T(x) = x + P^{-1}(b - Ax).
\]
Let \( x^{(0)} \in \mathbb{R}^n \) and define recursive \( x^{(m+1)} = T(x^{(m)}) \) for \( m \in \mathbb{N}_0 \). For each \( m \in \mathbb{N}_0 \) we define the \( m \)-th residual as
\[
r^{(m)} = b - Ax^{(m)}.
\]
Show that for all \( m \in \mathbb{N}_0 \) we have
\[
r^{(m+1)} := (I - AP^{-1})r^{(m)}.
\]

Exercise 8
Consider the matrix \( A \in \mathbb{R}^{2 \times 2} \) and vectors \( x, b \in \mathbb{R}^2 \) with \( Ax = b \), where
\[
A = \begin{pmatrix} 11 & 1 \\ -1 & 9 \end{pmatrix}, \quad x = \begin{pmatrix} 5 \\ -2 \end{pmatrix}, \quad b = \begin{pmatrix} 57 \\ -23 \end{pmatrix}.
\]
We guess that the diagonal matrix $P^{-1} \in \mathbb{R}^{2 \times 2}$ with

\[
P^{-1} = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}
\]

is a good approximate inverse of $A$. Conduct the iteration

\[
x^{(k+1)} = x^{(k)} + P^{-1}(b - Ax^{(k)})
\]

with starting value $x^{(0)} = 0$ for six steps and calculate $\|x - x^k\|$ at each step. Compute the eigenvalues of $I - P^{-1}A$ and compare with the ratio of the norms of consecutive errors $\|x - x^{k+1}\|/\|x - x^k\|$.

You can use Matlab or GNU Octave to solve this problem. If you don’t have access to Matlab or GNU Octave, you can use [https://octave-online.net/](https://octave-online.net/) for this exercise.