MATH 270A – NUMERICAL LINEAR ALGEBRA – HOMEWORK 3

Due Friday, November 9th. Handwritten submissions only.

Exercise 1
Given a vector $v \in \mathbb{R}^2$, we want to define a rotation matrix $G \in \mathbb{R}^{2\times 2}$ and $\gamma \in \mathbb{R}$ such that $Gv = \gamma e^1$. Write

$$G = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}.$$

Show that the following two constructions provide the coefficients c and s and the value γ.

- Variant 1:

$$\gamma = v_2 \cdot \nu, \quad s = \frac{1}{\nu}, \quad c := s\tau, \quad \tau = \frac{v_1}{v_2}, \quad \nu = \sqrt{1+\tau^2}.$$

- Variant 2:

$$\gamma = v_1 \cdot \nu, \quad c = \frac{1}{\nu}, \quad s := c\tau, \quad \tau = \frac{v_2}{v_1}, \quad \nu = \sqrt{1+\tau^2}.$$

Remark 1
In applications, the first variant is prefered if $|v_2| \geq |v_1|$ and the second variant is prefered if $|v_1| \geq |v_2|$.

Exercise 2
Let $A \in \mathbb{R}^{n\times m}$ with $\ker(A) = 0$ and let $b \in \mathbb{R}^n$. Show that the matrix

$$M = \begin{pmatrix} I & A \\ A^T & 0 \end{pmatrix} \tag{1}$$

is invertible. Prove that the solution of the system

$$\begin{pmatrix} I & A \\ A^T & 0 \end{pmatrix} \begin{pmatrix} r \\ x \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix} \tag{2}$$

satisfies $A^T A x = A^T b$.

Exercise 3
Let $A \in \mathbb{R}^{n\times m}$ with $\operatorname{ran}(A) = \mathbb{R}^n$ and let $b \in \mathbb{R}^n$. Show that the matrix

$$M = \begin{pmatrix} -I & A^T \\ A & 0 \end{pmatrix} \tag{3}$$

is invertible. Prove that the solution of the system

$$\begin{pmatrix} -I & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix} \tag{4}$$

satisfies $Ax = b$ with $x = A^T y$.

Exercise 4
Let $D \in \mathbb{R}^{n\times m}$ and $T \in \mathbb{R}^{m\times n}$ be matrices whose entries are zero except for the first d diagonal entries $D_{11}, D_{22}, \ldots, D_{dd}$ and $T_{11}, T_{22}, \ldots, T_{dd}$, respectively, where $d \leq \min(n, m)$. Assume that $T_{ii} = D_{ii}^{-1}$ for $1 \leq i \leq d$. Show that $T = D^\dagger$.

Exercise 5
Let $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{m\times n}$. Show that $B = A^\dagger$ if and only if the following four conditons hold:

$$ABA = A, \quad BAB = B, \quad (AB)^T = AB, \quad (BA)^T = BA.$$

Proof. Let $A = WDV^T$ be the (full) singular value decomposition of A. From the lecture and the previous exercise, we know that

$$A^\dagger = VD^\dagger W.$$

We verify that the pseudoinverse $A^\dagger$ satisfies the four required identities by simply plugging in the pseudoinverse and doing elementary computations.

We show that the pseudoinverse $A^\dagger$ is the only solution to the four equations

$$\begin{aligned} ABA &= A, \\ BAB &= B, \\ (AB)^T &= AB, \\ (BA)^T &= BA. \end{aligned}$$

One easily checks that $(A^T)^\dagger = (A^\dagger)^T$ using the singular value decomposition. Now we find

$$A^\dagger AB = A^\dagger ABAB = A^\dagger AA^T B^T B = A^T (A^\dagger)^T A^T B^T B = A^T B^T B = (BA)^T B = BAB = B$$

Taking the transpose of the first two conditions, we get $A^T B^T A^T = A^T$ and $B^T A^T B^T = B^T$. Now we find

$$A^\dagger AB = A^\dagger B^T A^T = A^\dagger AA^\dagger B^T A^T = A^\dagger A^{T\dagger} A^T B^T A^T = A^\dagger A^{T\dagger} A^T = A^\dagger AA^\dagger = A^\dagger.$$

This completes the proof. $\square$

Exercise 6
Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a contraction with contraction constant $q \in [0, 1)$. Let $x^* \in \mathbb{R}^n$ be the unique fixpoint of T. Let $x^{(0)} \in \mathbb{R}^n$ and define recursive $x^{(m+1)} = T(x^{(m)})$ for $m \in \mathbb{N}_0$.

Let $\epsilon > 0$. Find the minimum number M for which $\|x^{(M)} - x^*\| < \epsilon$ in terms of ϵ and

(a) the initial error norm $\|x^0 - x^*\|$.
(b) the initial difference $\|x^0 - x^1\|$.

Exercise 7
Consider the fixpoint iteration

$$T(x) = x + P^{-1}(b - Ax).$$

Let $x^{(0)} \in \mathbb{R}^n$ and define recursive $x^{(m+1)} = T(x^{(m)})$ for $m \in \mathbb{N}_0$. For each $m \in \mathbb{N}_0$ we define the m-th residual as

$$r^{(m)} = b - Ax^{(m)}.$$

Show that for all $m \in \mathbb{N}_0$ we have

$$r^{(m+1)} := (I - AP^{-1})r^{(m)}.$$

Exercise 8
Consider the matrix $A \in \mathbb{R}^{2\times 2}$ and vectors $x, b \in \mathbb{R}^2$ with $Ax = b$, where

$$A = \begin{pmatrix} 11 & -1 \\ -1 & 9 \end{pmatrix}, \quad x = \begin{pmatrix} 5 \\ -2 \end{pmatrix}, \quad b = \begin{pmatrix} 57 \\ -23 \end{pmatrix}. \tag{5}$$

We guess that the diagonal matrix $P^{-1} \in \mathbb{R}^{2\times 2}$ with

$$P^{-1} = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix} \tag{6}$$

is a good approximate inverse of A. Conduct the iteration

$$x^{(k+1)} = x^{(k)} + P^{-1}(b - Ax^{(k)}) \tag{7}$$

with starting value $x^{(0)} = 0$ for six steps and calculate $\|x - x^k\|$ at each step. Compute the eigenvalues of $I - P^{-1}A$ and compare with the ratio of the norms of consecutive errors $\|x - x^{k+1}\|/\|x - x^k\|$.

You can use Matlab or GNU Octave to solve this problem. If you don't have access to Matlab or GNU Octave, you can use `https://octave-online.net/` for this exercise.