MATH 270A – NUMERICAL LINEAR ALGEBRA – HOMEWORK 4

Due Monday, November 26th. Handwritten submissions only.

Exercise 1
Let $A \in \mathbb{R}^{n \times n}$. Show that there exists $\theta \in \mathbb{C}$ with $\mathrm{Id} - \theta^{-1} A$ being a contraction if and only if the eigenvalues of A are contained in the interior of a disk $D \subseteq \mathbb{C}$ that excludes the origin.

Remark: this completely characterizes for which parameters the Richardson iteration converges.

Solution 1
We check that the following are equivalent

- $\mathrm{Id} - \theta^{-1} A$ is a contraction.
- $\lambda(\mathrm{Id} - \theta^{-1} A)$ is contained in the open unit circle around the origin
- $\lambda(\theta^{-1} A - \mathrm{Id})$ is contained in the open unit circle around the origin
- $\lambda(\theta^{-1} A)$ is contained in the open unit circle around 1
- $\lambda(A)$ is contained in the open circle around θ of radius $|\theta|$.

The claim is now evident.

Exercise 2
Let $A \in \mathbb{R}^{n \times n}$ be symmetric positive definite. We can improve the convergence bound seen in the lecture.

(1) Suppose that the eigenvalues of A are contained the positive interval $[a, b] \subset \mathbb{R}^{+}$. Show that the Richardson iteration converges with parameter $\theta = \frac{1}{2}(b + a)$. Find a bound for the spectral radius of the iteration matrix in terms of a and b.
(2) Let $\lambda_{\min} > 0$ and $\lambda_{\max} > 0$ be the smallest and the largest eigenvalues of A, respectively. Show that the spectral radius of the Richardson method's iteration matrix is minimal among all $\theta > 0$ for $\theta = \frac{1}{2}(\lambda_{\max} + \lambda_{\min})$. Find a bound for the spectral radius of the iteration matrix in terms of $\lambda_{\min}$ and $\lambda_{\max}$.

Solution 2
The first statement is a consequence of the first exercise. The proof of the second statement is found in the lecture notes on the gradient method.

Exercise 3
Consider the rotation matrix

$$R = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}.$$

For which values of α does there exist $\theta > 0$ such that the Richardson method for R with parameter θ converges?

Solution 3
The characteristic polynomial of the rotation matrix is

$$\chi_R(\lambda) = (\cos(\alpha) - \lambda)^2 + \sin(\alpha) \tag{1}$$

$$= \lambda^2 - 2\cos(\alpha)\lambda + 1 \tag{2}$$

Hence we get $\chi_R(\lambda) = 0$ if and only if

$$(\lambda - \cos(\alpha))^2 = \cos(\alpha)^2 - 1 \iff \lambda - \cos(\alpha) = \pm \sin(\alpha)\mathrm{i} \iff \lambda = \cos(\alpha) \pm \sin(\alpha)\mathrm{i} \tag{3}$$

The two eigenvalues are contained in an open half-space within the complex plane if and only if α is neither $\pi/2$ nor $3\pi/2$. This precisely the case when the Richardson method has a parameter θ that makes the method converge, according to the first exercise.

Exercise 4
Let $A \in \mathbb{R}^{n\times n}$ be strictly diagonally dominant with diagonal part $D \in \mathbb{R}^{n\times n}$. Show that $\rho\left(I - D^{-1}A\right) < 1$, that is, the spectral radius of the Jacobi iteration matrix is a contraction.

Solution 4
One easily sees that for the matrix $\mathcal{H} = I - D^{-1}A$ we have

$$\sup_{x\in\mathbb{R}^n, x\neq 0} \frac{\|\mathcal{H}x\|_\infty}{\|x\|_\infty} < 1 \tag{4}$$

when A is strictly diagonally dominant, as follows from the characterization of the ∞-operator norm as the maximum row sum.

Exercise 5
Consider the matrix

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

(1) Determine the spectral radius of the Jacobian iteration matrix.
(2) Determine the spectral radius of the Gauss-Seidel iteration matrix.

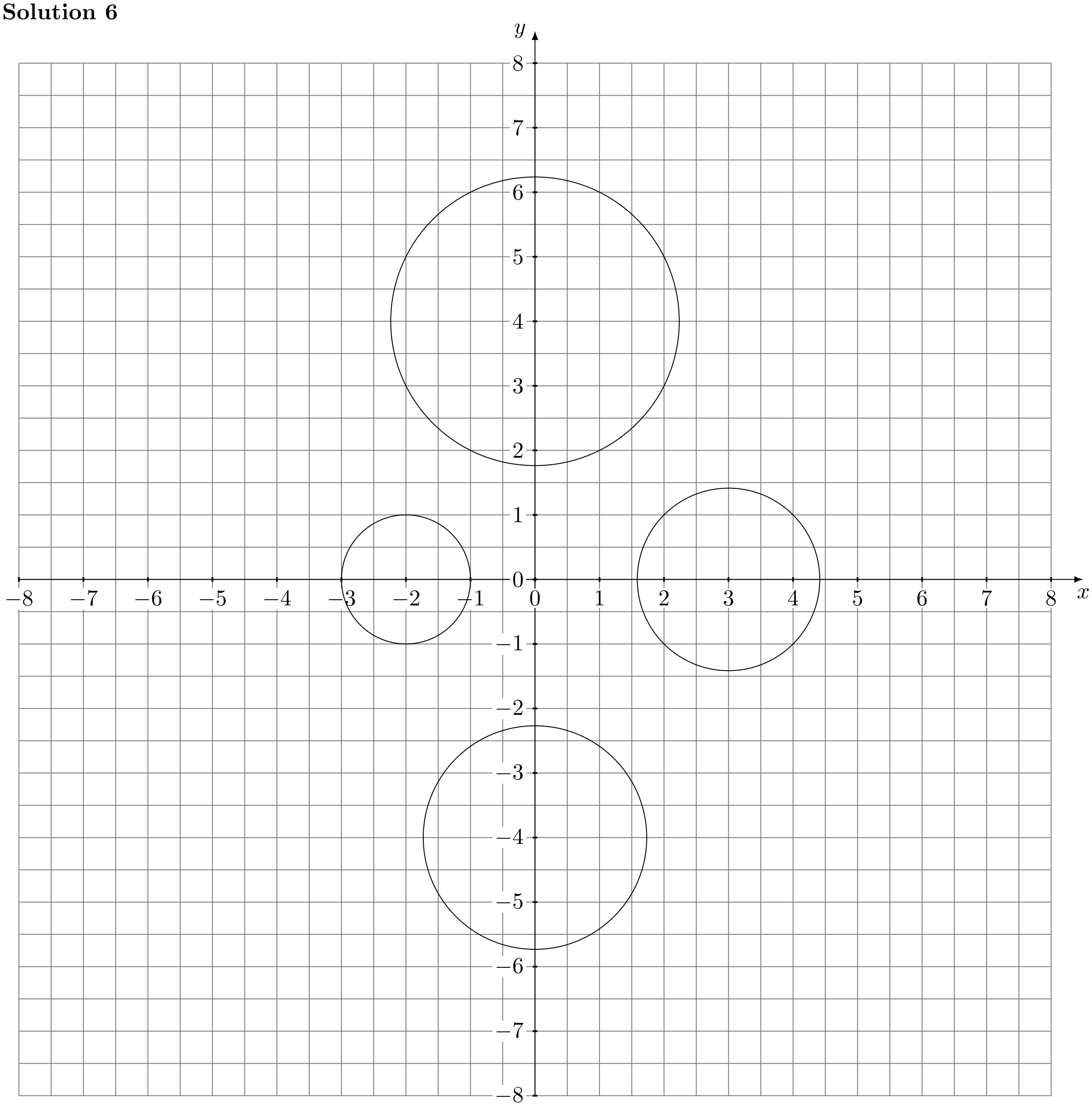
Solution 5
The spectral of the Jacobian iteration matrix is $\rho = 4.8025$. The spectral of the Gauss-Seidel iteration matrix is $\rho = 1.6667$.

Exercise 6
Consider the complex-valued matrix

$$A = \begin{pmatrix} 3 & 1 & \mathrm{i} & 0 \\ 1 & 4\mathrm{i} & 2 & 0 \\ 0 & 0 & -2 & -1 \\ 1 & 1 & 1 & -4\mathrm{i} \end{pmatrix}.$$

Draw the rowwise and columnwise Gerschgorin circles of this matrix and argue whether A is invertible.

Solution 6



Exercise 7
Suppose someone tries to implement the Richardson method with an implementation equivalent to the following pseudocode:

FOR $i = 1, \ldots, n$ DO
  $x_i = x_i + \theta^{-1} \left(b_i - \sum_{j=1}^{n} a_{ij} x_j \right)$
END FOR

Irrespective of whether that is a good idea or not, find the fixpoint method that is actually implemented. Explicitly state the preconditioner.

Solution 7
Given an iterate $x^{(k)}$, the next iterate $x^{(k+1)}$ satisfies

$$x_i^{(k+1)} = x_i^{(k)} + \theta^{-1} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i}^{n} a_{ij} x_j^{(k)} \right). \tag{5}$$

We rewrite this as

$$\theta x_i^{(k+1)} = y_i^{(k)} - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)}, \tag{6}$$

$$y_i^{(k)} := \theta x_i^{(k)} + b_i - \sum_{j=i}^{n} a_{ij} x_j^{(k)}, . \tag{7}$$

We recognize this is a forward substitution with

$$(L + \theta \, \mathrm{Id}) \, x^{(k+1)} = y^{(k)}, \quad y^{(k)} := b + \theta x^{(k)} - (D + R) x^{(k)}. \tag{8}$$

This corresponds to the matrix splitting

$$A = P + M, \quad P = L + \theta \, \mathrm{Id}, \quad M = (D + R) - \theta \, \mathrm{Id}\,. \tag{9}$$

Exercise 8
Let $A \in \mathbb{R}^{n \times n}$ be symmetric. Let $L, D, R \in \mathbb{R}^{n \times n}$ be the strictly lower triangular, diagonal, and strictly upper triangular parts of A, respectively. Consider the forward and backward SOR fixpoint mappings,

$$T_{SOR-L,\omega}(x) := \left(\omega^{-1} D + L\right)^{-1} (b - Rx - (1 - \omega^{-1}) Dx),$$
$$T_{SOR-R,\omega}(x) := \left(\omega^{-1} D + R\right)^{-1} (b - Lx - (1 - \omega^{-1}) Dx),$$

and define the SSOR fixpoint mapping,

$$T_{SSOR,\omega}(x) := T_{SOR-R,\omega} \left(T_{SOR-L,\omega}(x) \right).$$

Show that the preconditioner $P \in \mathbb{R}^{n \times n}$ of the SSOR method is given by

$$P = \frac{\omega}{2 - \omega} \left(\frac{D}{\omega} + L \right) D^{-1} \left(\frac{D}{\omega} + L^T \right)$$

and determine the difference $M = A - P$.

Solution 8
As in the lecture notes, we first find that

$$P^{-1} = \left(\left(\omega^{-1} D + L\right)^{-1} + \left(\omega^{-1} D + R\right)^{-1} - \left(\omega^{-1} D + R\right)^{-1} A \left(\omega^{-1} D + L\right)^{-1} \right).$$

Simple algebraic manipulations lead to

$$\begin{aligned}
P^{-1} &= \left(\left(\omega^{-1} D + L\right)^{-1} + \left(\omega^{-1} D + R\right)^{-1} - \left(\omega^{-1} D + R\right)^{-1} A \left(\omega^{-1} D + L\right)^{-1} \right) \\
&= \left(\omega^{-1} D + R\right)^{-1} \left(\left(\omega^{-1} D + R\right) \left(\omega^{-1} D + L\right)^{-1} + \mathrm{Id} - A \left(\omega^{-1} D + L\right)^{-1} \right) \\
&= \left(\omega^{-1} D + R\right)^{-1} \left(\left(\omega^{-1} D + R\right) + \left(\omega^{-1} D + L\right) - A \right) \left(\omega^{-1} D + L\right)^{-1} \\
&= \left(\omega^{-1} D + R\right)^{-1} \left(2\omega^{-1} D - D \right) \left(\omega^{-1} D + L\right)^{-1} \\
&= \frac{2 - \omega}{\omega} \left(\omega^{-1} D + R\right)^{-1} D \left(\omega^{-1} D + L\right)^{-1}
\end{aligned}$$

Inverting this expression and using symmetry gives the desired result.