MATH 270A – NUMERICAL LINEAR ALGEBRA – HOMEWORK 4

Due Monday, November 26th. Handwritten submissions only.

Exercise 1
Let $A \in \mathbb{R}^{n\times n}$. Show that there exists $\theta \in \mathbb{C}$ with $\operatorname{Id} - \theta^{-1}A$ being a contraction if and only if the eigenvalues of A are contained in the interior of a disk $D \subseteq \mathbb{C}$ that excludes the origin.

Remark: this completely characterizes for which parameters the Richardson iteration converges.

Exercise 2
Let $A \in \mathbb{R}^{n\times n}$ be symmetric positive definite. We can improve the convergence bound seen in the lecture.

(1) Suppose that the eigenvalues of A are contained the positive interval $[a, b] \subset \mathbb{R}^+$. Show that the Richardson iteration converges with parameter $\theta = \frac{1}{2}(b + a)$. Find a bound for the spectral radius of the iteration matrix in terms of a and b.
(2) Let $\lambda_{\min} > 0$ and $\lambda_{\max} > 0$ be the smallest and the largest eigenvalues of A, respectively. Show that the spectral radius of the Richardson method's iteration matrix is minimal among all $\theta > 0$ for $\theta = \frac{1}{2}(\lambda_{\max} + \lambda_{\min})$. Find a bound for the spectral radius of the iteration matrix in terms of $\lambda_{\min}$ and $\lambda_{\max}$.

Exercise 3
Consider the rotation matrix

$$R = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}.$$

For which values of α does there exist $\theta > 0$ such that the Richardson method for R with parameter θ converges?

Exercise 4
Let $A \in \mathbb{R}^{n\times n}$ be strictly diagonally dominant with diagonal part $D \in \mathbb{R}^{n\times n}$. Show that $\rho\left(I - D^{-1}A\right) < 1$, that is, the spectral radius of the Jacobi iteration matrix is a contraction.

Exercise 5
Consider the matrix

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

(1) Determine the spectral radius of the Jacobian iteration matrix.
(2) Determine the spectral radius of the Gauss-Seidel iteration matrix.

Exercise 6
Consider the complex-valued matrix

$$A = \begin{pmatrix} 3 & 1 & \mathrm{i} & 0 \\ 1 & 4\mathrm{i} & 2 & 0 \\ 0 & 0 & -2 & -1 \\ 1 & 1 & 1 & -4\mathrm{i} \end{pmatrix}.$$

Draw the rowwise and columnwise Gerschgorin circles of this matrix and argue whether A is invertible.

Exercise 7
Suppose someone tries to implement the Richardson method with an implementation equivalent to the following pseudocode:

FOR $i = 1, \dots, n$ DO
$x_i = x_i + \theta^{-1} \left(b_i - \sum_{j=1}^{n} a_{ij} x_j \right)$
END FOR

Irrespective of whether that is a good idea or not, find the fixpoint method that is actually implemented. Explicitly state the preconditioner.

Exercise 8
Let $A \in \mathbb{R}^{n \times n}$ be symmetric. Let $L, D, R \in \mathbb{R}^{n \times n}$ be the strictly lower triangular, diagonal, and strictly upper triangular parts of A, respectively. Consider the forward and backward SOR fixpoint mappings,

$$T_{SOR-L,\omega}(x) := \left(\omega^{-1} D + L\right)^{-1} (b - Rx - (1 - \omega^{-1}) Dx),$$
$$T_{SOR-R,\omega}(x) := \left(\omega^{-1} D + R\right)^{-1} (b - Lx - (1 - \omega^{-1}) Dx),$$

and define the SSOR fixpoint mapping,

$$T_{SSOR,\omega}(x) := T_{SOR-R,\omega} \left(T_{SOR-L,\omega}(x) \right).$$

Show that the preconditioner $P \in \mathbb{R}^{n \times n}$ of the SSOR method is given by

$$T_{SSOR,\omega}(x) = \frac{\omega}{2 - \omega} \left(\frac{D}{\omega} + L \right) D^{-1} \left(\frac{D}{\omega} + L^T \right)$$

and determine the difference $M = A - P$.