Exercise 1
Let \( A \in \mathbb{R}^{n \times n} \). Show that there exists \( \theta \in \mathbb{C} \) with \( \text{Id} - \theta^{-1}A \) being a contraction if and only if the eigenvalues of \( A \) are contained in the interior of a disk \( D \subseteq \mathbb{C} \) that excludes the origin.

Remark: this completely characterizes for which parameters the Richardson iteration converges.

Exercise 2
Let \( A \in \mathbb{R}^{n \times n} \) be symmetric positive definite. We can improve the convergence bound seen in the lecture.

(1) Suppose that the eigenvalues of \( A \) are contained the positive interval \([a, b] \subset \mathbb{R}^+\). Show that the Richardson iteration converges with parameter \( \theta = \frac{1}{2}(b + a) \). Find a bound for the spectral radius of the iteration matrix in terms of \( a \) and \( b \).

(2) Let \( \lambda_{\text{min}} > 0 \) and \( \lambda_{\text{max}} > 0 \) be the smallest and the largest eigenvalues of \( A \), respectively. Show that the spectral radius of the Richardson method’s iteration matrix is minimal among all \( \theta > 0 \) for \( \theta = \frac{1}{2}(\lambda_{\text{max}} + \lambda_{\text{min}}) \). Find a bound for the spectral radius of the iteration matrix in terms of \( \lambda_{\text{min}} \) and \( \lambda_{\text{max}} \).

Exercise 3
Consider the rotation matrix
\[
R = \begin{pmatrix}
\cos(\alpha) & -\sin(\alpha) \\
\sin(\alpha) & \cos(\alpha)
\end{pmatrix}.
\]

For which values of \( \alpha \) does there exist \( \theta > 0 \) such that the Richardson method for \( R \) with parameter \( \theta \) converges?

Exercise 4
Let \( A \in \mathbb{R}^{n \times n} \) be strictly diagonally dominant with diagonal part \( D \in \mathbb{R}^{n \times n} \). Show that \( \rho(I - D^{-1}A) < 1 \), that is, the spectral radius of the Jacobi iteration matrix is a contraction.

Exercise 5
Consider the matrix
\[
A = \begin{pmatrix}
3 & 1 & 1 \\
0 & 2 & 0 \\
1 & 0 & 2
\end{pmatrix}.
\]

(1) Determine the spectral radius of the Jacobian iteration matrix.
(2) Determine the spectral radius of the Gauss-Seidel iteration matrix.
Exercise 6
Consider the complex-valued matrix
\[
A = \begin{pmatrix}
3 & 1 & i & 0 \\
1 & 4i & 2 & 0 \\
0 & 0 & -2 & -1 \\
1 & 1 & 1 & -4i
\end{pmatrix}.
\]
Draw the rowwise and columnwise Gerschgorin circles of this matrix and argue whether \( A \) is invertible.

Exercise 7
Suppose someone tries to implement the Richardson method with an implementation equivalent to the following pseudocode:

\[
\text{FOR } i = 1, \ldots, n \text{ DO} \\
\quad x_i = x_i + \theta^{-1} \left( b_i - \sum_{j=1}^{n} a_{ij} x_j \right) \\
\text{END FOR}
\]
Irrespective of whether that is a good idea or not, find the fixpoint method that is actually implemented. Explicitly state the preconditioner.

Exercise 8
Let \( A \in \mathbb{R}^{n \times n} \) be symmetric. Consider the forward and backward SOR fixpoint mappings,
\[
T_{\text{SOR-}L,\omega}(x) := (\omega^{-1}D + L)^{-1}(b - Rx + (1 - \omega^{-1})Dx), \\
T_{\text{SOR-}R,\omega}(x) := (\omega^{-1}D + R)^{-1}(b - Lx + (1 - \omega^{-1})Dx),
\]
and define the SSOR fixpoint mapping,
\[
T_{\text{SSOR},\omega}(x) := T_{\text{SOR-}R,\omega}(T_{\text{SOR-}L,\omega}(x))
\]
Show that the preconditioner \( P \) of the SSOR method is given by
\[
T_{\text{SSOR},\omega}(x) = \frac{\omega}{2 - \omega} \left( \frac{D}{\omega} + L \right) D^{-1} \left( \frac{D}{\omega} + L^{T} \right)
\]
and determine the difference \( M = A - P \).