MATH 270A – NUMERICAL LINEAR ALGEBRA – HOMEWORK 5

Due Monday, December 3rd. Handwritten submissions only.

You may use computer software to do the calculation exercises and give rounded results.

Exercise 1
Consider the matrix $A \in \mathbb{R}^{4\times4}$ and the vector $b \in \mathbb{R}^4$ given by

$$A = \begin{pmatrix} 4 & 1 & 1 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 2 & 0 & 0 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} -4 \\ 1 \\ -5 \\ -8 \end{pmatrix}.$$

Use the starting vector $x^{(0)} = 0$ and conduct five steps each of the Jacobi iteration, the forward Gauss-Seidel iteration, and compute the Euclidean norms of the errors.

Solution 1
For the Jacobi method, the result is

step	approximate solution	error	error norm
0	$(0,0,0,0)^T$	$(-2,1,3,-1)^T$	3.8730
1	$(-1,0.5,2.5,-2)^T$	$(1,-0.5,-0.5,-1)^T$	1.5811
2	$(-1.75,1.25,2.75,-1.5)^T$	$(0.25,0.25,-0.25,0.5)^T$	0.66144
3	$(2,1,3.125,-1.125)^T$	$(0,0,0.125,-0.125)^T$	0.17678
4	$(2.0312,1.0625,3,-1)^T$	$(-0.03125,0.0625,0,0)^T$	0.069877
5	$(-2.01562,0.98438,3.03125,-0.98438)^T$	$(-0.015625,-0.015625,0.031250,0.015625)^T$	0.041340

For the forward Gauss-Seidel method, the result is

step	approximate solution	error	erro
0	$(0,0,0,0)^T$	$(-2,1,3,-1)^T$	3.873
1	$(-1,0,2.5,-1.5)^T$	$(1,-1,-0.5,-0.5)^T$	1.581
2	$(-1.625,0.9375,2.96875,-1.1875)^T$	$(0.375,-0.0625,-0.03125,-0.1875)^T$	0.425
3	$(-1.97656,0.99609,2.99805,-1.01172)^T$	$(0.0234375,-0.0039062,-0.0019531,-0.0117188)^T$	0.026
4	$(-1.99854,0.99976,2.99988,-1.00073)^T$	$\left(1.4648\times10^{-3},-2.4414\times10^{-4},-1.2207\times10^{-4},-7.3242\times10^{-4}\right)^T$	0.001
5	$(-1.99991,0.99998,2.99999,-1.00005)^T$	$\left(9.1553\times10^{-5},-1.5259\times10^{-5},-7.6294\times10^{-6},-4.5776\times10^{-5}\right)^T$	1.037

Exercise 2
Consider the symmetric positive definite matrix $A \in \mathbb{R}^{4\times4}$ and the vector $b \in \mathbb{R}^4$ given by

$$A = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 2 & 2 & 2 \\ -1 & 2 & 3 & 1 \\ -1 & 2 & 1 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} -1 \\ 1 \\ 6 \\ -7 \end{pmatrix}.$$

Use the starting vector $x^{(0)} = 0$ and conduct two steps of steepest descent algorithms. Compute the errors for each approximate solution.

Solution 2
We calculate

step	approximate solution	error	error norm
0	$(-0.18471, 0.18471, 1.10828, 1.29299)^T$	$(-0.81529, -27.184710, 14.891720, 9.707010)^T$	32.491
1	$(0.875390, -2.423590, 1.604010, 1.392140)^T$	$(-1.87540, -24.57640, 14.39600, 9.60790)^T$	30.118
2	$(0.633300, -2.287900, 2.629410, 2.423410)^T$	$(-1.6333, -24.7121, 13.3706, 8.5766)^T$	29.423
3	$(1.3137, -4.6306, 3.1690, 2.3549)$	$(-2.3137, -22.3694, 12.8310, 8.6451)$	27.297

Exercise 3
Consider the symmetric positive definite matrix $A \in \mathbb{R}^{3\times 3}$ and the vector $b \in \mathbb{R}^3$ given by

$$A = \begin{pmatrix} 4 & 1 & 2 \\ 1 & 9 & 1 \\ 2 & 1 & 16 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 18 \\ 16 \end{pmatrix}.$$

Use the starting vector $x^{(0)} = 0$ and conduct three steps of steepest descent algorithms with and without Jacobi preconditioning. Compute the errors for each approximate solution.

Solution 3
Without Jacobi preconditioning we calculate

step	approximate solution	error	error norm
0	$(0, 0, 0)^T$	$(-1, 2, 1)^T$	2
1	$(0.00000, 1.37586, 1.22298)^T$	$(-1.00000, 0.62414, -0.22298)^T$	1.1997
2	$(-0.35869, 1.78827, 0.75902)^T$	$(-0.64131, 0.21173, 0.24098)^T$	0.71707
3	$(-0.53897, 1.93327, 1.02728)$	$(-0.461025, 0.066729, -0.027282)$	0.46663

With Jacobi preconditiong we calculate

step	approximate solution	error	error norm
0	$(0, 0, 0)^T$	$(-1, 2, 1)^T$	2
1	$(0.00000, 1.85714, 0.92857)^T$	$(-1.000000, 0.142857, 0.071429)^T$	1.0127
2	$(-0.90562, 1.89584, 0.88503)^T$	$(-0.094380, 0.104155, 0.114968)^T$	0.18159
3	$(-0.91591, 1.99672, 0.98898)^T$	$(-0.0840907, 0.0032792, 0.0110238)^T$	0.084874

Exercise 4
Consider a symmetric positive definite matrix $A \in \mathbb{R}^{n\times n}$. Prove that the associated energy functional $\mathcal{J}$,

$$\mathcal{J}(x) = \frac{1}{2}\langle Ax, x\rangle - \langle b, x\rangle$$

is convex, that is,

$$\forall t \in [0,1] : \forall x, y \in \mathbb{R}^n : \mathcal{J}\left(tx + (1-t)y\right) \leq t\mathcal{J}\left(x\right) + (1-t)\mathcal{J}\left(y\right).$$

Solution 4
We calculate

$$\begin{aligned} \mathcal{J}\left(tx + (1-t)y\right) &= \frac{1}{2}\langle tAx + (1-t)Ay, tx + (1-t)y\rangle - \langle b, tx + (1-t)y\rangle \\ &= \frac{1}{2}t^2\langle Ax, x\rangle + (1-t)t\langle Ax, y\rangle + \frac{1}{2}(1-t)^2\langle Ay, y\rangle - t\langle b, x\rangle - (1-t)\langle b, y\rangle \end{aligned}$$

We calculate

$$\begin{aligned}
&\frac{1}{2}t\langle Ax,x\rangle + \frac{1}{2}(1-t)\langle Ay,y\rangle - \frac{1}{2}t^2\langle Ax,x\rangle - (1-t)t\langle Ax,y\rangle - \frac{1}{2}(1-t)^2\langle Ay,y\rangle \\
&= \frac{1}{2}(t-t^2)\langle Ax,x\rangle - (1-t)t\langle Ax,y\rangle + \frac{1}{2}\left((1-t)-(1-t)^2\right)\langle Ay,y\rangle \\
&= \frac{1}{2}t(1-t)\langle Ax,x\rangle - (1-t)t\langle Ax,y\rangle + \frac{1}{2}t(1-t)\langle Ay,y\rangle \\
&= \frac{1}{2}t(1-t)\langle A(x-y),(x-y)\rangle \\
&\geq 0.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\mathcal{J}\left(tx+(1-t)y\right) &\leq \frac{1}{2}t\langle Ax,x\rangle + \frac{1}{2}(1-t)\langle Ay,y\rangle - t\langle b,x\rangle - (1-t)\langle b,y\rangle \\
&= t\mathcal{J}\left(x\right) + (1-t)\mathcal{J}\left(y\right).
\end{aligned}$$

This completes the proof.