Exercise 1 (Sparse Linear Algebra and Richardson Iteration)
When we save every entry of an \( n \times n \) matrix \( A \) explicitly, then we use \( n^2 \) entries in memory. If the vast majority of entries in \( A \) is zero, then more memory-efficient formats are possible.

The idea is to describe the non-zero structure of a matrix and its entries. Describing the non-zero structure structure explicitly incurs a little overhead, but the savings in terms of memory are dramatic if the number of non-zero entries is much less than \( n^2 \). Furthermore, knowing the non-zero structure of a matrix can dramatically speed up an algorithm.

In the following exposition and exercises we use zero-based indexing for arrays, as in C, Java, or Python, but unlike Fortran or Matlab, which use one-based indexing. However, you can translate the idea into any indexing-convention of your choice.

(1) The coordinate format saves a matrix in the following form: two arrays of indices \( R \) and \( C \) and an array of floating-point numbers \( V \), each of length \( \ell \), where \( \ell \) is the number of non-zero entries of \( A \). For the \( i \)-th entry, the matrix has a non-zero value \( V[\ell] \) in the entry at row \( R[i] \) and column \( C[i] \).

For example, the matrix

\[
A = \begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & \\
8 & 9 & 6 & 7 \\
10 \\
\end{pmatrix}
\]

would be described in a C data structure as

\[
\text{int } R[10] = \{ 0, 0, 0, 1, 1, 2, 2, 3, 3, 5 \}; \\
\text{int } C[10] = \{ 0, 1, 2, 1, 2, 4, 5, 0, 2, 4 \}; \\
\text{int } V[10] = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \};
\]

Write a method that on input \( \ell \), \( R \), \( C \), \( V \), and \( x \in \mathbb{R}^n \) outputs the product \( Ax \). You may use the above matrix as an example.

(2) We generate a class of matrices in coordinate format that appears in practical applications. Let \( m \in \mathbb{N} \) and \( n = m^2 \). We define a matrix \( A \) as follows. The diagonal entries of \( A \) have the value \( 4n \). Furthermore, for each row with index \( k = i \cdot m + j \) with \( 0 \leq i, j < m \), we have the off-diagonal entries at column indices

\[
\{ (i+1) \cdot m + j, \ (i-1) \cdot m + j, \ i \cdot m + j + 1, \ i \cdot m + j - 1 \} \cap \{0, \ldots, n-1\}
\]

being set to \(-n^2\).

Write an algorithm that produces a matrix of this form in coordinate format.

(3) The compressed sparse row format is even more, well, compressed. Letting \( n \) denote the number of rows of the matrix \( A \in \mathbb{R}^{n \times n} \), we let \( R \) be an array of integers of length \( n + 1 \), and we let \( C \) be an array of integers and \( V \) be an array of floating-point numbers, both of which having length \( V[n] \). Their significance is as follows.

The last entry of \( V \) is the number of non-zero entries in the matrix \( A \). Similar as with the coordinate format, the array \( C \) saves the column index of each entry, and the array \( V \) saves the numerical value of each entry. The array \( R \), however, is read differently from the Coordinate format.
For any $0 \leq i < n$, every index $R[i] \leq k < R[i+1]$ corresponds to a non-zero entry at row $i$ and column $C[k]$ with value $V[k]$.

For instance, the matrix $A$ used as an example above would be described in a C data structure as

```c
int R[7] = { 0, 3, 5, 7, 9, 9, 10 };
int C[10] = { 0, 1, 2, 1, 2, 4, 5, 0, 2, 4 };
int V[10] = { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 };
```

Write a method that on input $n$, $R$, $C$, $V$, and $x$ outputs the product $Ax$. You may use the above matrix as an example.

(4) In practice, it is convenient to assemble a matrix first in coordinate format and then convert it into compressed sparse row format for faster matrix multiplication. Write an algorithm that transforms a matrix in coordinate format into a matrix in compressed sparse row format.

(5) A sparse matrix $A$ arising in numerical partial differential equations may have $k$ non-zero entries per row. Suppose that integers/indices are saved with 4 bytes each and that floating-point values occupy 8 bytes. If matrix has size $N \times N$, how much memory does it occupy for each of the previous three formats? Give an example with $N = 10^3, 10^6, 10^9$ and state with decimal (or binary) prefixes (Kilo,Mega,Tera,...).

Remark 1
The sparse matrix formats require some overhead to describe the non-zero structure of the matrix. This overhead pays off in many applications because the number of non-zero entries is magnitudes smaller than the number of total entries. For example, for $N \times N$ matrices in finite element methods, the number of non-zero entries grows only linearly in $N$ instead of quadratically.

Different sparse matrix formats have different advantages and disadvantages. The coordinate format occupies a bit more memory than the compressed sparse rows format, but it is rather easy to append (or erase) matrix entries. A matrix in compressed sparse row format is less easy to modify, but the format is rather compact.

A major advantage of the compressed sparse row format is that the matrix-vector product can be parallelized over the rows, which gives a considerable speed up on many-core CPUs.