Thesis Abstract – Martin Werner Licht

My PhD thesis

"On the A Priori and A Posteriori Error Analysis in Finite Element Exterior Calculus"

elaborates upon the theoretical foundations of finite element methods through the perspective of *Finite Element Exterior Calculus* (FEEC). This framework has been popularized in recent years by Douglas Arnold, Richard Falk, and Ragnar Winther in a series of publications. FEEC unifies many numerical methods for partial differential equations in the language of differential geometry.

The fundamental insight of finite element exterior calculus is that *differential complexes* are not only fundamental for the theoretical analysis of many prominent partial differential equations, such as in electromagnetism, elasticity, and relativity, but should also be the groundwork in developing numerical methods. For example, the solution theory for the Hodge-Laplace equation is best understood by studying variations of the de Rham complex. Accordingly, *finite element de Rham complexes* are the starting point in developing practically useful numerical methods.

The concept of "differential complex" has become a staple in the numerical literature largely due to the impact of finite element exterior calculus. The framework uses the unifying language of differential forms, thus building a bridge between numerical analysis on the one hand and functional analysis, global analysis, and algebraic topology on the other hand. The approach of replicating the de Rham complex as an underlying qualitative structure on the numerical level makes finite element exterior calculus an instance of a *structure-preserving discretization.*

My PhD thesis gives a systematic exposition of this important topic and addresses fundamental open questions. One major aspect is the construction of *commuting bounded projections.* Generally speaking, I am interested in commuting diagrams of the form

$$
\begin{array}{ccccccc}
H\Lambda^0(\Omega) & \xrightarrow{\mathrm{d}} & H\Lambda^1(\Omega) & \xrightarrow{\mathrm{d}} & \dots & \xrightarrow{\mathrm{d}} & H\Lambda^n(\Omega) \\
\downarrow{\scriptstyle \pi^0} & & \downarrow{\scriptstyle \pi^1} & & & & \downarrow{\scriptstyle \pi^n} \\
\mathcal{P}\Lambda^0(\mathcal{T}) & \xrightarrow{\mathrm{d}} & \mathcal{P}\Lambda^1(\mathcal{T}) & \xrightarrow{\mathrm{d}} & \dots & \xrightarrow{\mathrm{d}} & \mathcal{P}\Lambda^n(\mathcal{T}).
\end{array}
$$

Here, the first row is the *Sobolev de Rham complex* over a domain Ω, and the second row is a *finite element de Rham complex* over a triangulation $\mathcal{T}$ of that domain, constituting of spaces of piecewise polynomial differential forms. The vertical arrows are projections that commute with the exterior derivative and are uniformly bounded in the discretization parameters, such as the mesh size. In three dimensions, the diagram can be instantiated into the more popular form

$$
\begin{array}{ccccccc}
H^1(\Omega) & \xrightarrow{\mathrm{grad}} & H(\mathrm{curl},\Omega) & \xrightarrow{\mathrm{curl}} & H(\mathrm{div},\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) \\
\downarrow{\scriptstyle \pi^0} & & \downarrow{\scriptstyle \pi^1} & & \downarrow{\scriptstyle \pi^2} & & \downarrow{\scriptstyle \pi^3} \\
\mathbf{P}_1(\mathcal{T}) & \xrightarrow{\mathrm{grad}} & \mathbf{Ned}(\mathcal{T}) & \xrightarrow{\mathrm{curl}} & \mathbf{RT}(\mathcal{T}) & \xrightarrow{\mathrm{div}} & \mathbf{P}_{0,\mathrm{DC}}\Lambda^n(\mathcal{T}).
\end{array}
$$

The Sobolev spaces in the first row are well-known in the theory of mathematical electromagnetism, as are the finite element spaces in the second row: the first order Lagrange space, the lowest-order Nédélec space and Raviart-Thomas space, and piecewise constant functions.

The commuting bounded projections π^k are critically important in relating the discrete theory with the analytical theory: given a commuting projection, the stability and convergence of numerical methods are well-established in the literature.

A major part of my PhD thesis and published research concerns the construction of commuting projections in non-smooth but practically relevant settings. The construction of commuting bounded projections in rough geometric settings is very intricate and utilizes a variety of tools from different mathematical areas, such as Lipschitz topology, geometric measure theory, and functional analysis.

As one main contribution, I address numerical analysis over the class of domains known as *weakly Lipschitz domains*. Those are domains whose boundary can be locally flattened by a bi-Lipschitz transformation. This is strictly more general than the well-known class of Lipschitz domains. In the absence of any higher regularity, I use the collar theorem in the Lipschitz category to establish the existence of a tubular neighborhood along the boundary of such domains. My application is the construction of extension operators for Sobolev differential forms that commute with the exterior derivatives.

Another major contribution is my analysis of numerical methods for the Hodge-Laplace equation with *mixed boundary conditions*. Partial differential equations in vector fields with mixed boundary conditions are an emergent topic. My PhD thesis establishes that de Rham complexes with partial boundary conditions (representing what is also known as "essential boundary conditions" in the calculus of variations) are the right starting point for their theoretical and numerical analysis. While one easily establishes finite element de Rham complexes with partial boundary conditions, the construction of a commuting bounded projection is a challenge overcome in my PhD thesis. This has extended finite element exterior calculus to mixed boundary conditions.

The existence of a smoothed projection in rough geometric settings enables the stability and convergence of finite element methods for partial differential equations, also known as *a priori error analysis*. The subsequent parts of my PhD thesis address the *a posteriori error analysis* in finite element exterior calculus.

The idea of a posteriori error analysis is that, once a numerical approximation has been computed, we can practically compute upper bounds for the approximation error even if the true solution is not known. My PhD thesis develops tools for such error analysis in finite element exterior calculus using the duality gap between variational formulations and their dual formulation, in this context also known as *Prager-Synge* theorem.

For that purpose, I develop the notion of *discrete distributional differential form*. Differential complexes of such forms generalize finite element differential complexes and introduce new ways of computing their homology theory and Poincaré-Friedrichs constants. These innovative tools relate to algebraic concepts such as *double complexes* and *homology with local coefficients*. By the time of this writing, those concepts and techniques have found unexpected applications in recent work on mixed-dimensional partial differential equations.

The coda of my PhD thesis is the construction of *equilibrated a posteriori error estimators* for the curl-curl equation. These error estimators are easy to compute and offer tighter error bounds than more classical error estimators. Their construction was previously only understood in the case of lowest-order finite elements but is now also possible for higher-order finite element methods.