MATH 174/274
NUMERICAL METHODS FOR PHYSICAL MODELING
HOMEWORK 5

Exercise 1
Give an example of a continuous function $f : (0, 1) \to \mathbb{R}$ does not have *any* upper bound.

Exercise 2
Give an example of a continuous function $g : (0, 1) \to \mathbb{R}$ that does have a least upper bound but does not attain that upper bound anywhere over $(0, 1)$.

Exercise 3
Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Show that there exist $y, z \in [a, b]$ such that for all $x \in [a, b]$ we have $f(y) \leq f(x) \leq f(z)$.

Exercise 4
Perform five steps of the bisection method to find a root of $f(x) = x^3 - 2$ with starting interval $[a_0, b_0] = [0, 2]$.

Exercise 5
Perform three steps of the Newton method to find a root of $f(x) = x^3 - 2$ with starting value $x_0 = 2$. Write down the iteration formula.

Exercise 6
Describe how the Newton method could be used to approximate solution of the following nonlinear system of equations in the unknowns x and y:

$$x^3 y = e^{xy}, \quad \sin(xy) = y^2.$$

State the iteration formula of the Newton method.

Exercise 7
Can you state the iteration formula of the Newton method to solve the linear system of equations $Ax = b$, where $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$? Discuss whether this approach is practically useful or not.

Exercise 8
Suppose you have a continuous function $f : [0, 1] \to \mathbb{R}$. Besides continuity, you know that f assumes a positive value and a negative value *somewhere*. Describe an algoritm that is guaranteed to find a positive and a negative value of f. Your algorithm can probe $f(x)$ for any $x \in [0, 1]$ but it must be deterministic. The runtime of the algorithm can be arbitrarily large but it must be guaranteed to finish.

Remark: this shows that if we know that a continuous functions over an interval has got a negative and a positive value, then we reliably find the starting interval for the bisection method.