

ON IRREGULAR COLORINGS OF GRAPHS

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Abstract

For a graph G and a proper coloring $c : V(G) \rightarrow \{1, 2, \dots, k\}$ of the vertices of G for some positive integer k , the color code of a vertex v of G (with respect to c) is the ordered $(k + 1)$ -tuple $\text{code}(v) = (a_0, a_1, \dots, a_k)$ where a_0 is the color assigned to v and for $1 \leq i \leq k$, a_i is the number of vertices adjacent to v that are colored i . The coloring c is irregular if distinct vertices have distinct color codes and the irregular chromatic number $\chi_{ir}(G)$ of G is the minimum positive integer k for which G has an irregular k -coloring. We establish sharp upper and lower bounds for the irregular chromatic number of a disconnected graph in terms of the irregular chromatic numbers of its components. Irregular chromatic numbers of some classes of disconnected graphs are determined. It is shown that if G is a nontrivial graph of order n , then $2\sqrt{n} \leq \chi_{ir}(G) + \chi_{ir}(\overline{G}) \leq 2n$, $n \leq \chi_{ir}(G)\chi_{ir}(\overline{G}) \leq n^2$, and each bound in these inequalities is sharp.

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1. Introduction

A (proper) *coloring* of a graph G is a function $c : V(G) \rightarrow \mathbf{N}$ having the property that $c(u) \neq c(v)$ for every pair u, v of adjacent vertices of G , where \mathbf{N} is the set of positive integers. A k -*coloring* of G uses k colors. The *chromatic number* $\chi(G)$ of G is the minimum positive integer k for which there is a k -coloring of G . For a positive integer k and a proper coloring $c : V(G) \rightarrow \{1, 2, \dots, k\}$ of the vertices of a graph G , the *color code* of a vertex v of G (with respect to c) is the ordered $(k + 1)$ -tuple

$$\text{code}_c(v) = (a_0, a_1, \dots, a_k),$$

where a_0 is the color assigned to v (that is, $a_0 = c(v)$) and for $1 \leq i \leq k$, a_i is the number of vertices adjacent to v that are colored i . If the coloring c is clear, we write $\text{code}_c(v)$ as

$\text{code}(v)$ or simply, $\text{code}(v) = a_0a_1a_2 \cdots a_k$. Therefore, if $a_0 = i$, then $a_i = 0$, for $1 \leq i \leq k$ and $\sum_{i=1}^k a_i = \deg_G v$. The coloring c is called *irregular* if distinct vertices have distinct color codes and the *irregular chromatic number* $\chi_{ir}(G)$ of G is the minimum positive integer k for which G has an irregular k -coloring. An irregular k -coloring with $\chi_{ir}(G) = k$ is a *minimum irregular coloring*. Since every irregular coloring of a graph G is a coloring of G , it follows that $\chi(G) \leq \chi_{ir}(G)$. The concept of irregular coloring was introduced and studied in [3], inspired by the problem in graph theory that concerns finding means to distinguish all the vertices of a connected graph.

To illustrate this concept, consider the Petersen graph P of Figure 1. Since $\chi(P) = 3$, it follows that $\chi_{ir}(P) \geq 3$. A 4-coloring of the Petersen graph is given in Figure 1 along with the corresponding codes of its vertices. Since distinct vertices have distinct codes, this coloring is irregular and so $\chi_{ir}(P) \leq 4$. Therefore, $\chi_{ir}(P) = 3$ or $\chi_{ir}(P) = 4$. In fact, $\chi_{ir}(P) = 4$, as we now establish. Assume, to the contrary, that $\chi_{ir}(P) = 3$ and let c be an irregular 3-coloring of P . By the Pigeonhole Principle, at least four vertices of P must be colored the same by c . We may assume, without loss of generality, that there are four vertices colored 1 by c . Thus the color codes for these four vertices are 1012, 1021, 1030, 1003. Let u and v be the vertices of P such that $\text{code}(u) = 1030$ and $\text{code}(v) = 1003$. Thus u is adjacent to three vertices colored 2 and v is adjacent to three vertices colored 3. Since $\text{diam}(P) = 2$, there is a $u - v$ path of length 2 in P , say u, w, v is such a path in P . However then, $c(u) = 2$ and $c(w) = 3$, which is impossible. Thus c is not irregular. Therefore, $\chi_{ir}(P) = 4$.

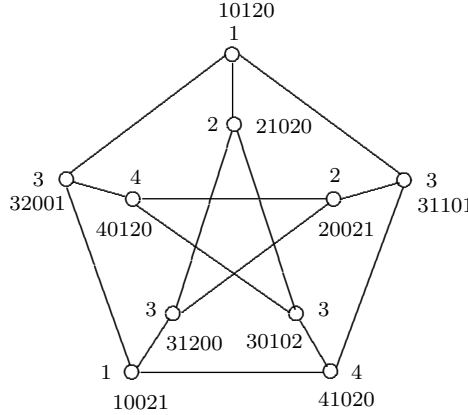


Figure 1: An irregular coloring of the Petersen graph

The *neighborhood* of a vertex u in a graph G is $N(u) = \{v \in V(G) : uv \in E(G)\}$. The following useful observation was stated in [3].

Observation 1.1. *Let c be a (proper) coloring of the vertices of a nontrivial graph G and let u and v be two distinct vertices of G .*

- (a) *If $c(u) \neq c(v)$, then $\text{code}(u) \neq \text{code}(v)$.*
- (b) *If $\deg_G u \neq \deg_G v$, then $\text{code}(u) \neq \text{code}(v)$.*
- (c) *If c is irregular and $N(u) = N(v)$, then $c(u) \neq c(v)$.*

Among the results established in [3] are the following three theorems.

Theorem 1.2. *For every pair a, b of integers with $2 \leq a \leq b$, there is a connected graph G with $\chi(G) = a$ and $\chi_{ir}(G) = b$.*

Theorem 1.3. *If a nontrivial connected graph G has an irregular k -coloring, then G contains at most $k \binom{r+k-2}{r}$ vertices of degree r .*

Since every nontrivial graph G has at least two vertices of the same degree, any irregular coloring of G must use at least two distinct colors. Furthermore, the coloring of a graph G that assigns distinct colors to distinct vertices of G is irregular and so $\chi_{ir}(G)$ always exists. Therefore, if G is a nontrivial graph of order n , then $2 \leq \chi_{ir}(G) \leq n$. For each even integer $n = 2k$, where k is a positive integer, let F_n be the bipartite graph with partite sets $X = \{x_1, x_2, \dots, x_k\}$ and $Y = \{y_1, y_2, \dots, y_k\}$ such that $\deg x_i = \deg y_i = i$ for $1 \leq i \leq k$.

Theorem 1.4. *Let G be a nontrivial connected graph of order n . Then*

- (a) $\chi_{ir}(G) = 2$ if and only if n is even and $G \cong F_n$;
- (b) $\chi_{ir}(G) = n$ if and only if G is a complete multipartite graph.

Furthermore, for each pair k, n of integers with $2 \leq k \leq n$, there exists a connected graph of order n having irregular chromatic number k if and only if $(k, n) \neq (2, n)$ for some odd integer n .

In this work, we study the relationship between the irregular chromatic number of a disconnected graph and that of its components in Section 2. For a disconnected graph G with components G_1, G_2, \dots, G_p , we establish sharp upper and lower bounds for $\chi_{ir}(G)$ in terms of $\chi_{ir}(G_i)$ for $1 \leq i \leq p$. With this information, we are able to obtain Nordhaus-Gaddum inequalities for the irregular chromatic number of a graph in Section 3 and verify the sharpness of each bound in these inequalities. Irregular chromatic numbers of some classes of disconnected graphs are determined in Section 4. We refer to the book [2] for graph theory notation and terminology not described in this paper.

2. Irregular Colorings of Disconnected Graphs

Let G_1, G_2, \dots, G_p be $p \geq 2$ graphs and let $G = G_1 \cup G_2 \cup \dots \cup G_p$ be the union of these p graphs, that is, $\{V(G_1), V(G_2), \dots, V(G_p)\}$ is a partition of $V(G)$ and $E(G) = E(G_1) \cup E(G_2) \cup \dots \cup E(G_p)$. If each $G_i \cong F$ for some graph F , then we write $G = pF$. In order to establish upper and lower bounds for the irregular chromatic number of G in terms of the irregular chromatic numbers of the graphs G_i ($1 \leq i \leq p$), we first present some preliminary results. For a coloring c of G , the coloring c_i of G_i ($1 \leq i \leq p$) defined by $c_i(v) = c(v)$ for all $v \in V(G_i)$ is said to be induced by c . Then $\text{code}_c(v) = \text{code}_{c_i}(v)$ for all $v \in V(G_i)$ where $1 \leq i \leq p$. This observation gives us the following result.

Lemma 2.1. *Let G_1, G_2, \dots, G_p be $p \geq 2$ graphs and let c be an irregular coloring of $G_1 \cup G_2 \cup \dots \cup G_p$. For each integer i with $1 \leq i \leq p$, the coloring c_i of G_i induced by c is irregular.*

On the other hand, suppose that c_i is a coloring of a graph G_i for $1 \leq i \leq p$. The coloring c of $G = G_1 \cup G_2 \cup \dots \cup G_p$ defined by $c(v) = c_i(v)$ for all $v \in V(G_i)$ is called the *extension* (or

extended coloring) of the colorings c_i ($1 \leq i \leq p$). Then $\text{code}_c(v) = \text{code}_{c_i}(v)$ for all $v \in V(G_i)$, where $1 \leq i \leq p$. Note that if c_i is an irregular coloring of G_i for $1 \leq i \leq p$, then the extension c of the colorings c_i ($1 \leq i \leq p$) need not be irregular. However, under additional conditions, c is irregular. For a coloring c of a graph G , the *color set* of c is the set $\{c(v) : v \in V(G)\}$.

Lemma 2.2. *Let G_1, G_2, \dots, G_p be $p \geq 2$ graphs and let c_i be an irregular coloring of G_i for $1 \leq i \leq p$. If the color sets of c_i ($1 \leq i \leq p$) are pairwise disjoint, then the extended coloring c of $G = G_1 \cup G_2 \cup \dots \cup G_p$ is irregular.*

With the aid of Lemmas 2.1 and 2.2, we obtain the following result.

Theorem 2.3. *Let G_1, G_2, \dots, G_p be $p \geq 2$ graphs and let $G = G_1 \cup G_2 \cup \dots \cup G_p$. Then*

$$\max\{\chi_{ir}(G_i) : 1 \leq i \leq p\} \leq \chi_{ir}(G) \leq \sum_{i=1}^p \chi_{ir}(G_i).$$

Both the lower and upper bounds in Theorem 2.3 are sharp. For a graph G , the *degree set* of G is the set $\{\deg v : v \in V(G)\}$.

Proposition 2.4. *Let G_1, G_2, \dots, G_p be $p \geq 2$ graphs and let $G = G_1 \cup G_2 \cup \dots \cup G_p$. If the degree sets of the graphs G_i ($1 \leq i \leq p$) are pairwise disjoint, then*

$$\chi_{ir}(G) = \max\{\chi_{ir}(G_i) : 1 \leq i \leq p\}.$$

The proof of Proposition 2.4 is straightforward and is therefore omitted. By Proposition 2.4, the lower bound in Theorem 2.3 is sharp. We now consider the upper bound in Theorem 2.3. First, we show that if no graph G_i ($1 \leq i \leq p$) contains isolated vertices, then the upper bound in Theorem 2.3 cannot be sharp.

Theorem 2.5. *If G_1, G_2, \dots, G_p are p graphs without isolated vertices, then*

$$\chi_{ir}(G_1 \cup G_2 \cup \dots \cup G_p) \leq \left(\sum_{i=1}^p \chi_{ir}(G_i) \right) - p + 1.$$

Proof. Certainly, the result is true for $p = 1$. Thus, we may assume that $p \geq 2$. We proceed by induction on p . First, we establish the result for $p = 2$. Let G_1 and G_2 be two graphs without isolated vertices such that $\chi_{ir}(G_i) = k_i$ for $i = 1, 2$. Suppose that c_1 is an irregular k_1 -coloring of G_1 using colors in $\{1, 2, \dots, k_1\}$ and c_2 is an irregular k_2 -coloring of G_2 using colors in $\{k_1, k_1 + 1, \dots, k_1 + k_2 - 1\}$. Extend the colorings c_1 and c_2 to the $(k_1 + k_2 - 1)$ -coloring c of $G_1 \cup G_2$. We show that c is irregular. Let $x, y \in V(G_1 \cup G_2)$. We may assume that $x \in V(G_1)$ and $y \in V(G_2)$. If $c(x) \neq c(y)$, then $\text{code}(x) \neq \text{code}(y)$. Assume therefore that $c(x) = c(y) = k_1$. Since x is not an isolated vertex of G_1 , it follows that x is adjacent to some vertex z in G_1 . Thus $c(z) \neq k_1$ and so $c(z) \leq k_1 - 1$. Since y is not adjacent to any vertex colored $c(z)$, it follows that $\text{code}(x) \neq \text{code}(y)$. Therefore, c is an irregular $(k_1 + k_2 - 1)$ -coloring of $G_1 \cup G_2$ and so

$$\chi_{ir}(G_1 \cup G_2) \leq k_1 + k_2 - 1 = \chi_{ir}(G_1) + \chi_{ir}(G_2) - 1.$$

Consequently, the result holds for $p = 2$.

Suppose that the result holds for any $p - 1 \geq 2$ graphs without isolated vertices. Let G_1, G_2, \dots, G_p be p graphs without isolated vertices, and let $H_1 = G_1 \cup G_2 \cup \dots \cup G_{p-1}$ and $H_2 = G_p$. The proof then can be completed by applying the induction hypothesis to H_1 and H_2 . \square

Proposition 2.6. *If G_1 and G_2 are two graphs such that exactly one of G_1 and G_2 contains isolated vertices, then*

$$\chi_{ir}(G_1 \cup G_2) \leq \chi_{ir}(G_1) + \chi_{ir}(G_2) - 1.$$

Proof. Suppose that G_1 contains isolated vertices and G_2 does not contain isolated vertices. Let V_1^* be the set of isolated vertices of G_1 . If G_1 is not an empty graph, let $G_1^* = G_1 - V_1^*$, $F_1 = \langle V_1^* \rangle$, and $F_2 = G_1^* \cup G_2$; otherwise, let $F_2 = G_2$. Then $G_1 \cup G_2 = F_1 \cup F_2$. Since the degree sets of F_1 and F_2 are disjoint, it follows by Proposition 2.4 that

$$\chi_{ir}(G_1 \cup G_2) = \max\{\chi_{ir}(F_1), \chi_{ir}(F_2)\}. \quad (1)$$

If G_1 is an empty graph, then $F_2 = G_2$ and so

$$\chi_{ir}(F_2) = \chi_{ir}(G_2) \leq \chi_{ir}(G_1) + \chi_{ir}(G_2) - 1.$$

If G_1 is not an empty graph, then $F_2 = G_1^* \cup G_2$. Since neither G_1^* nor G_2 contains isolated vertices, it follows by Theorem 2.5 that

$$\chi_{ir}(F_2) \leq \chi_{ir}(G_1^*) + \chi_{ir}(G_2) - 1 \leq \chi_{ir}(G_1) + \chi_{ir}(G_2) - 1.$$

Furthermore, $\chi_{ir}(F_1) = |V_1^*| \leq \chi_{ir}(G_1) \leq \chi_{ir}(G_1) + \chi_{ir}(G_2) - 1$. It then follows by (1) that $\chi_{ir}(G_1 \cup G_2) \leq \chi_{ir}(G_1) + \chi_{ir}(G_2) - 1$. \square

Theorem 2.7. *Let G_1, G_2, \dots, G_p be $p \geq 2$ graphs. Then*

$$\chi_{ir}(G_1 \cup G_2 \cup \dots \cup G_p) = \sum_{i=1}^p \chi_{ir}(G_i)$$

if and only if each G_i contains exactly $\chi_{ir}(G_i)$ isolated vertices for all i with $1 \leq i \leq p$.

Proof. We proceed by induction on p . First, we establish the result for $p = 2$. Let G_1 and G_2 be two graphs, where $\chi_{ir}(G_i) = k_i$ for $i = 1, 2$ and let $G = G_1 \cup G_2$. For $i = 1, 2$, let V_i^* be the set of isolated vertices of G_i . If $V_i^* \neq V(G_i)$ ($i = 1, 2$), then let $G_i^* = G_i - V_i^*$. In this case, since $G_i = G_i^* \cup \langle V_i^* \rangle$, it follows by Theorem 2.3 that $\chi_{ir}(G_i^*) \leq \chi_{ir}(G_i)$. We show that $\chi_{ir}(G) = k_1 + k_2$ if and only if $|V_i^*| = k_i$ for $i = 1, 2$.

Suppose that $|V_i^*| = k_i$ for $i = 1, 2$. We show that $\chi_{ir}(G) = k_1 + k_2$. If $V_i^* = V(G_i)$ for $i = 1, 2$, then the result is immediate. Thus we may assume that at least one of G_1 and G_2 is not an empty graph. Thus at least one of G_1^* and G_2^* exists. Suppose that exactly one of G_1 and G_2 is not an empty graph, say G_1 is not an empty graph. Then

$$\chi_{ir}(G_1^*) \leq \chi_{ir}(G_1) = k_1 \leq k_1 + k_2 - 1. \quad (2)$$

Otherwise, neither G_1 nor G_2 is an empty graph. Since G_i^* ($i = 1, 2$) contains no isolated vertices, it follows by Theorem 2.5 that

$$\chi_{ir}(G_1^* \cup G_2^*) \leq \chi_{ir}(G_1^*) + \chi_{ir}(G_2^*) - 1 \leq k_1 + k_2 - 1. \quad (3)$$

On the other hand, $\langle V_1^* \cup V_2^* \rangle$ is an empty graph of order $k_1 + k_2$ and so

$$\chi_{ir}(\langle V_1^* \cup V_2^* \rangle) = k_1 + k_2. \quad (4)$$

Let $G^* = G_1^*$ if G_1 is not an empty graph and G_2 is an empty graph; while let $G^* = G_1^* \cup G_2^*$ if neither G_1 nor G_2 is an empty graph. Since the degree sets of G^* and $\langle V_1^* \cup V_2^* \rangle$ are disjoint and $G = G^* \cup \langle V_1^* \cup V_2^* \rangle$, it then follows by (3), (4), and Proposition 2.4 that

$$\chi_{ir}(G) = \max\{\chi_{ir}(G^*), \chi_{ir}(\langle V_1^* \cup V_2^* \rangle)\} = k_1 + k_2.$$

For the converse, suppose that $G = G_1 \cup G_2$ such that $\chi_{ir}(G) = k_1 + k_2$. We show that $|V_i^*| = k_i$ for $i = 1, 2$. If at most one of the sets V_i^* ($i = 1, 2$) is not empty, then $\chi_{ir}(G) \leq k_1 + k_2 - 1$ by Theorem 2.5 and Proposition 2.6, which is a contradiction. Thus $V_i^* \neq \emptyset$ for $i = 1, 2$. By Theorem 2.3,

$$\chi_{ir}(G) \geq \chi_{ir}(\langle V_1^* \cup V_2^* \rangle) = |V_1^* \cup V_2^*|.$$

Since $\chi_{ir}(G) = k_1 + k_2$, it follows that $|V_1^* \cup V_2^*| \leq k_1 + k_2$. If $|V_1^* \cup V_2^*| \leq k_1 + k_2 - 1$, then

$$\chi_{ir}(\langle V_1^* \cup V_2^* \rangle) \leq k_1 + k_2 - 1. \quad (5)$$

It then follows by (2), (3), (5), and Proposition 2.4, that

$$\chi_{ir}(G) = \max\{\chi_{ir}(G^*), \chi_{ir}(\langle V_1^* \cup V_2^* \rangle)\} \leq k_1 + k_2 - 1,$$

which contradicts $\chi_{ir}(G) = k_1 + k_2$. Thus $|V_1^* \cup V_2^*| = k_1 + k_2$. Furthermore, $k_i = \chi_{ir}(G_i) \geq \chi_{ir}(\langle V_i^* \rangle) = |V_i^*|$ and so $|V_i^*| = k_i$ for $i = 1, 2$. Thus the result holds for $p = 2$.

Suppose that the result holds for any $p - 1 \geq 2$ graphs. Let G_1, G_2, \dots, G_p be $p \geq 3$ graphs. Let $H_1 = G_1 \cup G_2 \cup \dots \cup G_{p-1}$ and $H_2 = G_p$. Furthermore, let V_i^* be the set of isolated vertices of G_i ($1 \leq i \leq p$). Then $U_1^* = V_1^* \cup V_2^* \dots \cup V_{p-1}^*$ is the set of isolated vertices of H_1 and $U_2^* = V_p^*$ is the set of isolated vertices of $H_2 = G_p$. Since the result holds for two graphs, $\chi_{ir}(H_1 \cup H_2) = \chi_{ir}(H_1) + \chi_{ir}(H_2)$ if and only if $|U_1^*| = \chi_{ir}(H_1)$ and $|V_p^*| = |U_1^*| = \chi_{ir}(H_2) = k_p$. By the induction hypothesis, $\chi_{ir}(H_1) = \sum_{i=1}^{p-1} \chi_{ir}(G_i)$ if and only if $|V_i^*| = k_i$ for $1 \leq i \leq p - 1$. Combining this two statements, we obtain $\chi_{ir}(G) = \sum_{i=1}^p \chi_{ir}(G_i)$ if and only if $|V_i^*| = k_i$ for $1 \leq i \leq p$, as desired. \square

3. Nordhaus-Gaddum Inequalities

Probably the best known result involving a graph and its complement provides upper and lower bounds for both the sum and the product of the chromatic numbers of a graph and its complement. The following theorem is due to Nordhaus-Gaddum [1].

Theorem 3.1. For every graph G of order n ,

$$\begin{aligned} 2\sqrt{n} &\leq \chi(G) + \chi(\overline{G}) \leq n + 1 \\ n &\leq \chi(G)\chi(\overline{G}) \leq \left(\frac{n+1}{2}\right)^2. \end{aligned}$$

Indeed for any graphical parameter f and any graph G of order n , sharp upper and lower bounds for both $f(G) + f(\overline{G})$ and $f(G) \cdot f(\overline{G})$ are referred to as *Nordhaus-Gaddum inequalities*.

For an integer $n \geq 2$, let \mathcal{G}_n denote the set of graphs (connected or disconnected) of order n . We have seen that $2 \leq \chi_{ir}(G) \leq n$ and $\chi_{ir}(G) \geq \chi(G)$ for every nontrivial graph G of order n . These observations together with Theorem 3.1 yield the following Nordhaus-Gaddum inequalities for the irregular chromatic number of a graph.

Theorem 3.2. If G is a graph of order n , then

$$\begin{aligned} 2\sqrt{n} &\leq \chi_{ir}(G) + \chi_{ir}(\overline{G}) \leq 2n \\ n &\leq \chi_{ir}(G)\chi_{ir}(\overline{G}) \leq n^2. \end{aligned}$$

Each of the four bounds in Theorem 3.2 is sharp. In fact, more can be said.

Proposition 3.3. Let $G \in \mathcal{G}_n$, where $n \geq 1$. Then $\chi_{ir}(G) + \chi_{ir}(\overline{G}) = 2n$ (or $\chi_{ir}(G)\chi_{ir}(\overline{G}) = n^2$) if and only if $G = K_n$ or $G = \overline{K}_n$.

Proof. Since $\chi_{ir}(K_n) = \chi_{ir}(\overline{K}_n) = n$, it follows that $\chi_{ir}(G) + \chi_{ir}(\overline{G}) = 2n$ if $G = K_n$ or $G = \overline{K}_n$. Thus it remains only to verify the converse. Assume that $G \in \mathcal{G}_n$ such that $\chi_{ir}(G) + \chi_{ir}(\overline{G}) = 2n$. Then $\chi_{ir}(G) = \chi_{ir}(\overline{G}) = n$. Since at least one of G and \overline{G} is connected, we assume, without loss of generality, that G is connected.

Assume, to the contrary, that $G \neq K_n$. By Theorem 1.4, $G = K_{n_1, n_2, \dots, n_k}$, where $2 \leq k < n$, $n = n_1 + n_2 + \dots + n_k$, and $n_1 \leq n_2 \leq \dots \leq n_k$. Thus $\overline{G} = K_{n_1} \cup K_{n_2} \cup \dots \cup K_{n_k}$. If $n_1 \geq 2$, then each graph K_{n_i} has no isolated vertices for $1 \leq i \leq k$. Since $k \geq 2$, it follows by Theorem 2.5 that

$$\chi_{ir}(\overline{G}) \leq \left(\sum_{i=1}^k \chi_{ir}(K_{n_i}) \right) - k + 1 = \left(\sum_{i=1}^k n_i \right) - k + 1 \leq n - 1,$$

which is a contradiction. Thus we may assume that $n_1 = 1$. Let i be the smallest integer with $2 \leq i \leq k$ such that $n_i \geq 2$. Then $\overline{G} = (i-1)K_1 \cup K_{n_i} \cup K_{n_{i+1}} \cup \dots \cup K_{n_k}$. Let $H_1 = (i-1)K_1$ and $H_2 = K_{n_i} \cup K_{n_{i+1}} \cup \dots \cup K_{n_k}$. Then $\chi_{ir}(H_1) = i-1 < n-1$. Since each K_{n_j} ($i \leq j \leq k$) contains no isolated vertices, it follows by Theorem 2.5 that

$$\begin{aligned} \chi_{ir}(H_2) &\leq \left(\sum_{j=i}^k \chi_{ir}(K_{n_j}) \right) - (k-i+1) + 1 \\ &= \left(\sum_{j=i}^k n_j \right) - (k-i) \leq \sum_{j=i}^k n_j < n-1. \end{aligned}$$

Since $G = H_1 \cup H_2$ and the degree sets of H_1 and H_2 are disjoint, it follows by Theorem 2.4 that

$$\chi_{ir}(G) = \max\{\chi_{ir}(H_1), \chi_{ir}(H_2)\} < n - 1,$$

which is a contradiction. \square

Next, we establish the sharpness of the two lower bounds in Theorem 3.2, beginning with $\chi_{ir}(G)\chi_{ir}(\overline{G}) \geq n$ for every graph G of order n . First, recall that for each even integer $n = 2k$, where k is a positive integer, the graph F_n is the bipartite graph with partite sets $X = \{x_1, x_2, \dots, x_k\}$ and $Y = \{y_1, y_2, \dots, y_k\}$ such that $\deg x_i = \deg y_i = i$ for $1 \leq i \leq k$. Observe then that x_k is adjacent to all vertices in Y and x_{k-1} is adjacent to all vertices in $Y - \{y_1\}$. In general, for each integer j with $1 \leq j \leq k$,

$$N(x_j) = \{y_i : k - j + 1 \leq i \leq k\}.$$

By Theorem 1.4(a), the graph F_n is the only connected graph of order n with irregular chromatic number 2.

Proposition 3.4. *For every positive even integer n , there exists a connected graph G of order n such that $\chi_{ir}(G)\chi_{ir}(\overline{G}) = n$.*

Proof. Let $n = 2k$ for some positive integer k . We show that for each positive integer k , $\chi_{ir}(\overline{F}_{2k}) = k$. Since the result is obvious for $k = 1$, we assume that $k \geq 2$. Since \overline{F}_{2k} contains K_k as a subgraph, $\chi_{ir}(\overline{F}_{2k}) \geq k$. It remains only to show that $\chi_{ir}(\overline{F}_{2k}) \leq k$. Let $X = \{x_1, x_2, \dots, x_k\}$ and $Y = \{y_1, y_2, \dots, y_k\}$ be the partite sets of F_{2k} , where $\deg_{F_{2k}} x_i = \deg_{F_{2k}} y_i = i$ for $1 \leq i \leq k$. Thus for each integer j with $1 \leq j \leq k$, the neighborhood of x_j in F_{2k} is

$$N_{F_{2k}}(x_j) = \{y_i : k - j + 1 \leq i \leq k\}.$$

Therefore, in \overline{F}_{2k} ,

$$\deg_{\overline{F}_{2k}} x_i = \deg_{\overline{F}_{2k}} y_i = 2k - 1 - i$$

for $1 \leq i \leq k$ and for each integer j with $1 \leq j \leq k$, the neighborhood of x_j in \overline{F}_{2k} is

$$N_{\overline{F}_{2k}}(x_j) = \{y_i : 1 \leq i \leq k - j\} \cup (X - \{x_j\}).$$

Define a coloring c on \overline{F}_{2k} by

$$c(x_i) = i \text{ and } c(y_i) = k + 1 - i$$

for $1 \leq i \leq k$. Observe that if $c(u) = c(v) = i$, then $\{u, v\} = \{x_i, y_{k+1-i}\}$ for some i with $1 \leq i \leq k$. Since x_i is adjacent to y_{k+1-i} in F_{2k} , it follows that x_i is not adjacent to y_{k+1-i} in \overline{F}_{2k} . Thus c is a proper coloring of \overline{F}_{2k} .

Next, we show that c is irregular. Let u and v be two vertices of the same degree in \overline{F}_{2k} . Thus $u = x_i$ and $v = y_i$ for some i with $1 \leq i \leq k$. Then $c(x_i) = i$ and $c(y_i) = k + 1 - i$. If $i \neq k + 1 - i$, then $c(x_i) \neq c(y_i)$ and so x_i and y_i have different codes. Thus we may assume that $i = k + 1 - i$ and so $i = (k + 1)/2 \geq 2$. Observe that x_i is adjacent to y_1 in \overline{F}_{2k} and y_i is adjacent to x_1 in \overline{F}_{2k} . Since x_1 is colored 1 and y_1 is colored k by c , it follows that x_i is adjacent to only one vertex colored 1 in \overline{F}_{2k} , namely x_1 ; while y_1 is adjacent to two vertices colored 1 in \overline{F}_{2k} , namely x_1 and y_k . Thus x_i and y_i have different codes. Therefore, c is an irregular coloring of \overline{F}_{2k} with k colors, implying that $\chi_{ir}(\overline{F}_{2k}) \leq k$. \square

Next, we show that the lower bound in the inequality $\chi_{ir}(G) + \chi_{ir}(\overline{G}) \geq 2\sqrt{n}$ in Theorem 3.2 is sharp. Observe that if G is a nontrivial graph of order n such that $\chi_{ir}(G) + \chi_{ir}(\overline{G}) = 2\sqrt{n}$, then obviously $n = p^2$ for some positive integer p .

Theorem 3.5. *For every positive integer p , there exists a graph G of order $n = p^2$ such that $\chi_{ir}(G) + \chi_{ir}(\overline{G}) = 2\sqrt{n}$.*

Proof. For $p = n = 1$, let $G = K_1$, while for $p = 2$ and $n = 4$, let $G = P_4$. Since G is self-complementary and $\chi_{ir}(G) = p$ in both instances, the result holds for $p = 1, 2$. For $p = 3$ and $n = 9$, let G be the graph of order 9 in Figure 2, where minimum irregular 3-colorings are given for G and \overline{G} . Thus $\chi_{ir}(G) = \chi_{ir}(\overline{G}) = 3$. Therefore, there exists a graph G of order n such that $\chi_{ir}(G) + \chi_{ir}(\overline{G}) = 2\sqrt{n}$ for $n = 2^2$ and $n = 3^2$.

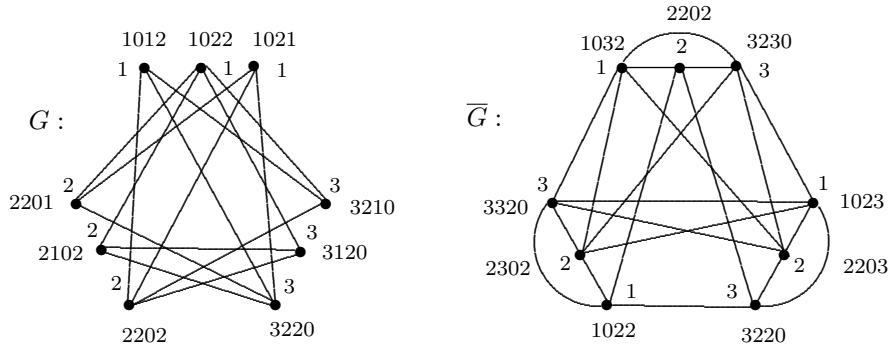


Figure 2: Graphs G and \overline{G} of order 9 with $\chi_{ir}(G) + \chi_{ir}(\overline{G}) = 6$

We now assume that $n = p^2$, where $p \geq 4$. For $1 \leq j \leq p$, let

$$\begin{aligned} U_j &= \{v_{i,j+(i-1)} : 1 \leq i \leq p\} = \{v_{1,j}, v_{2,j+1}, v_{3,j+2}, \dots, v_{p,j+(p-1)}\} \\ V_j &= \{v_{j,i} : 1 \leq i \leq p\} = \{v_{j,1}, v_{j,2}, \dots, v_{j,p}\} \\ W_j &= \{v_{i,j} : 1 \leq i \leq p\} = \{v_{1,j}, v_{2,j}, v_{3,j}, \dots, v_{p,j}\}, \end{aligned}$$

where the subscripts are expressed as the integers $1, 2, \dots, p$ modulo p . Now let

$$V(G) = \{v_{i,j} : 1 \leq i, j \leq p\} = \bigcup_{j=1}^p U_j = \bigcup_{j=1}^p V_j = \bigcup_{j=1}^p W_j.$$

Furthermore, let F_j be a complete graph on the set U_j for $1 \leq j \leq p$. We consider two cases, according to whether p is even or p is odd.

Case i. p is even.

For $1 \leq j \leq p$, construct a $(j-1)$ -regular graph G_j on the set W_j . Define

$$E(G) = \bigcup_{j=1}^p (E(F_j) \cup E(G_j)).$$

Thus, for each integer j with $1 \leq j \leq p$,

- (i) U_j is independent in \overline{G} and V_j is independent in G ,
- (ii) if $v_{i,j+(i-1)} \in U_j$, where $1 \leq i \leq p$, then $\deg_G v_{i,j+(i-1)} = p + i - 2$; if $v_{j,i} \in V_j$, where $1 \leq i \leq p$, then $\deg_G v_{j,i} = p + i - 2$. Thus all vertices in U_j have different degrees and all vertices in V_j have different degrees in G (and so in \overline{G}).

First, we show that $\chi_{ir}(G) = p$. Since G contains K_p as a subgraph, $\chi_{ir}(G) \geq p$. By (i), we can define a p -coloring c of G by $c(v) = j$ if $v \in V_j$ for $1 \leq j \leq p$. By (ii), this p -coloring c is irregular and so $\chi_{ir}(G) \leq p$. Hence $\chi_{ir}(G) = p$. Next, we show that $\chi_{ir}(\overline{G}) = p$. Since each subgraph induced by V_j in \overline{G} is a complete graph of order p , it follows that $\chi_{ir}(\overline{G}) \geq p$. To show that $\chi_{ir}(\overline{G}) \leq p$, we can define a p -coloring \overline{c} by $\overline{c}(u) = j$ if $u \in U_j$ for $1 \leq j \leq p$. By (i) and (ii), \overline{c} is irregular and so $\chi_{ir}(\overline{G}) \leq p$. Hence $\chi_{ir}(\overline{G}) = p$.

Case ii. p is odd.

Since $p \equiv 1 \pmod{4}$ or $p \equiv 3 \pmod{4}$, it follows that $p = 4k + 1$ or $p = 4k + 3$ for some positive integer k . For $1 \leq j \leq 2k$, construct a $(2j)$ -regular graph H_{2j} on the set W_{2j} and a $(2j)$ -regular graph H_{2j+1} on the set W_{2j+1} . For $1 \leq j \leq k$, let I_j be the 1-factor whose vertex set is $W_{4j-2} \cup W_{4j}$ and whose edge set is

$$E(I_j) = \{v_{i,4j-2}v_{i+1,4j} : 1 \leq i \leq p\}.$$

Now let F , H , and I be graphs, each of whose vertex sets is $V(G)$ and such that

$$\begin{aligned} E(F) &= \bigcup_{j=1}^p E(F_j) \\ E(H) &= \bigcup_{j=1}^{2k} (E(H_{2j}) \cup E(H_{2j+1})) \\ E(I) &= \bigcup_{j=1}^k E(I_j). \end{aligned}$$

Furthermore, let

$$X = E(F) \cup E(H) \cup E(I).$$

We now consider two subcases.

Subcase 2.1. $p \equiv 1 \pmod{4}$ and so $p = 4k + 1$.

Let $E(G) = X$. Thus for each integer j with $1 \leq j \leq p$, U_j is independent in \overline{G} and V_j is independent in G . Observe that for each integer j with $1 \leq j \leq p$, the set U_j consists of p vertices, exactly one of which belongs to each set W_j for $1 \leq j \leq p$. Thus all vertices in U_j have distinct degrees. Similarly, for each integer j with $1 \leq j \leq p$, the set V_j consists of p vertices, exactly one of which belongs to each set W_j for $1 \leq j \leq p$ and so all vertices in V_j have distinct degrees.

We now show that $\chi_{ir}(G) = \chi_{ir}(\overline{G}) = p$, beginning with $\chi_{ir}(G) = p$. Since G contains K_p as a subgraph, $\chi_{ir}(G) \geq p$. Since V_j is independent in G , we can define a p -coloring c of G by $c(v) = j$ if $v \in V_j$ for $1 \leq j \leq p$. Since the degrees of the vertices in each set V_j ($1 \leq j \leq p$) are distinct, the p -coloring c is irregular and so $\chi_{ir}(G) \leq p$. Hence $\chi_{ir}(G) = p$. Next, we show that $\chi_{ir}(\overline{G}) = p$. Since each subgraph induced by V_j in \overline{G} is a complete graph of order p , it

follows that $\chi_{ir}(\overline{G}) \geq p$. We define a p -coloring \overline{c} on \overline{G} by $\overline{c}(u) = j$ if $u \in U_j$ for $1 \leq j \leq p$. Since \overline{c} is irregular, $\chi_{ir}(\overline{G}) \leq p$. Hence $\chi_{ir}(\overline{G}) = p$. Therefore, $\chi_{ir}(G) = \chi_{ir}(\overline{G}) = p$.

Subcase 2.2. $p \equiv 3 \pmod{4}$ and so $p = 4k + 3$.

Let I_0 be a 1-factor whose vertex set is $W_1 \cup W_{4k+2}$ and whose edge set is

$$E(I_0) = \{v_{i,4k+2}v_{i+1,1} : 1 \leq i \leq p\}.$$

Now let $E(G) = X \cup E(I_0)$. Again for each integer j with $1 \leq j \leq p$, U_j is independent in \overline{G} and V_j is independent in G . An argument similar to that used in Subcase 2.1 shows that for each integer j with $1 \leq j \leq p$, all vertices in U_j have distinct degrees and all vertices in V_j have distinct degrees in G (and so in \overline{G}). Also, using the p -colorings c and \overline{c} described in Subcase 2.1, we obtain the minimum irregular p -colorings of G and \overline{G} , respectively. Therefore, $\chi_{ir}(G) = \chi_{ir}(\overline{G}) = p$. \square

A point (a, b) in the plane is called a *lattice point* if a and b are integers. If G is a graph of order n such that $\chi_{ir}(G) = a$ and $\chi_{ir}(\overline{G}) = b$, then $2\sqrt{n} \leq a + b \leq 2n$ and $n \leq ab \leq n^2$, which is equivalent to $n \leq ab$ and $a + b \leq 2n$. For this reason, we define a lattice point (a, b) to be *realizable* with respect to an integer n if $n \leq ab$ and $a + b \leq 2n$ and there is a graph G of order n such that $\chi_{ir}(G) = a$ and $\chi_{ir}(\overline{G}) = b$. For the chromatic number, Stewart [4] showed that no improvement in Theorem 3.1 is possible (without employing additional conditions). We state Stewart's theorem as follows.

Theorem 3.6. *Let n be a positive integer. For every two integers a and b such that*

$$2\sqrt{n} \leq a + b \leq n + 1, \text{ and } n \leq ab \leq \left(\frac{n+1}{2}\right)^2$$

there is a graph G of order n such that $\chi(G) = a$ and $\chi(\overline{G}) = b$.

There is, however, no corresponding result for irregular chromatic numbers. In order to show this, we first characterize all nontrivial graphs G of order n with $\chi_{ir}(G) = 2$.

Proposition 3.7. *Let G be a graph of order $n \geq 2$. Then $\chi_{ir}(G) = 2$ if and only if (1) n is even and $G \in \{2K_1, F_n, 2K_1 \cup F_{n-2}\}$, or (2) n is odd and $G = K_1 \cup F_{n-1}$.*

Proof. It is easy to verify that $\chi_{ir}(G) = 2$ for each of the graphs stated above. It remains to verify the converse. If G is connected, then $G \cong F_n$, where n is even by Theorem 1.4. Thus, we may assume that G is disconnected. Then $G = G_1 \cup G_2 \cup \cdots \cup G_k$, where $k \geq 2$ and G_i is a connected graph of order n_i for $1 \leq i \leq k$. By Theorem 2.3, $\chi_{ir}(G_i) \leq \chi_{ir}(G) = 2$ for $1 \leq i \leq k$. Thus, if G_i ($1 \leq i \leq k$) is a nontrivial component of G , then $\chi_{ir}(G_i) = 2$. We may assume, without loss of generality, that $n_1 \leq n_2 \leq \cdots \leq n_k$. If $n_1 \geq 2$, then $\chi_{ir}(G_i) = 2$ for $1 \leq i \leq k$. Thus $G_i \cong F_{n_i}$, where $n_i \geq 2$ is an even integer for $1 \leq i \leq k$. Since $k \geq 2$, it follows that G contains at least four end-vertices. On the other hand, since $\chi_{ir}(G) = 2$, it follows by Theorem 1.3 that G contains at most two end-vertices, which is impossible. Thus we may assume that $n_1 = 1$. Furthermore, G contains exactly one nontrivial component and this nontrivial component has irregular chromatic number 2. Let i be the smallest integer with $2 \leq i \leq k$ such that $n_i \geq 2$. If $i \geq 4$, then G contains three isolated vertices and so $\chi_{ir}(G) \geq 3$, which is impossible. Thus $i = 2$ or $i = 3$. Thus G either has exactly one isolated vertex or has exactly two isolated vertices. Therefore, if G has exactly one isolated vertex, then $G = K_1 \cup F_{n-1}$ where

n is odd; while if G has exactly two isolated vertices, then either $G = 2K_1$ or $G = 2K_1 \cup F_{n-2}$ for an even integer $n \geq 4$. \square

We have seen in the proof of Proposition 3.4 that $\chi_{ir}(\overline{F}_n) = \frac{n}{2}$ for every even integer $n \geq 2$. Furthermore, observe that $\chi_{ir}(\overline{2K_1}) = 2$, $\chi_{ir}(\overline{2K_1 \cup F_{n-2}}) = \frac{n}{2} + 1$ for every even integer $n \geq 2$ and $\chi_{ir}(\overline{K_1 \cup F_{n-1}}) = \frac{n+1}{2}$ for each odd integer $n \geq 3$. Therefore, the following is an immediate consequence of Proposition 3.7.

Corollary 3.8. *If G is a graph of order $n \geq 2$ with $\chi_{ir}(G) = 2$, then*

$$\chi_{ir}(\overline{G}) = \begin{cases} \frac{n}{2} \text{ or } \frac{n+2}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

By Corollary 3.8, if a, b , and $n \geq 2$ are positive integers such that one of a and b is 2 but the other is none of $\frac{n}{2}$, $\frac{n+1}{2}$, or $\frac{n+2}{2}$, then (a, b) is not realizable with respect to n . We illustrate this for $n = 9$, in which case every lattice point (a, b) is realizable for $b \geq 3$, while a lattice point $(a, 2)$ is realizable if and only if $a = 5$ by Corollary 3.8. Since $\overline{G} = G$ for every graph G , if a lattice point (a, b) is realizable, then so too is (b, a) . Thus, we may assume that $a \geq b$. For each realizable lattice point (a, b) , a graph G with $(\chi_{ir}(G), \chi_{ir}(\overline{G})) = (a, b)$ is shown as follows:

(9, 9)	K_9	(9, 8)	$K_{1,8}$	(9, 7)	$K_{2,7}$
(9, 6)	$K_{3,6}$	(9, 5)	$K_{4,5}$	(9, 4)	$K_{2,3,4}$
(9, 3)	$K_{2,2,2,3}$	(8, 8)	$K_{1,7} \cup K_1$	(8, 7)	$K_{2,6} \cup K_1$
(8, 6)	$K_{3,5} \cup K_1$	(8, 5)	$K_{4,4} \cup K_1$	(8, 4)	$K_{1,1,2,2,2} \cup K_1$
(8, 3)	G_1 in Figure 3	(7, 7)	$K_{2,5} \cup K_2$	(7, 6)	$K_{3,4} \cup 2K_1$
(7, 5)	$K_{2,2,3} \cup 2K_1$	(7, 4)	$\overline{G} = K_3 \cup 2P_3$	(7, 3)	$C_6 + K_3$
(6, 6)	$K_{2,2,2} \cup K_3$	(6, 5)	$\overline{G} = K_{2,3} \cup K_{1,2} \cup K_1$	(6, 4)	$\overline{G} = (2P_3 \cup 2K_1) + K_1$
(6, 3)	$\overline{G} = P_7 \cup P_2$	(5, 5)	$K_{2,3} \cup 2K_2$	(5, 4)	$(2P_3 \cup K_1) + 2K_1$
(5, 3)	$\overline{G} = P_6 \cup P_3$	(5, 2)	$\overline{G} = F_8 \cup K_1$	(4, 4)	G_2 in Figure 3
(4, 3)	G_3 in Figure 3	(3, 3)	Figure 2		

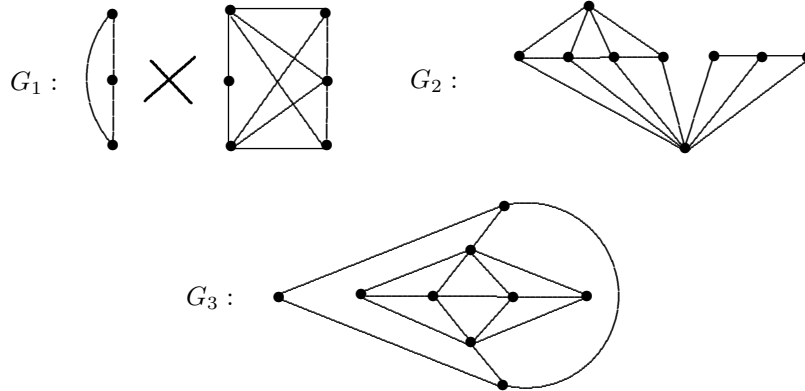


Figure 3: Graphs G_1 , G_2 and G_3

We conclude this section with the following question.

Problem 3.9. *Let $n \geq 2$ be an integer. For which lattice points (a, b) with $a, b \geq 2$, $n \leq ab$, and $a + b \leq 2n$, does there exist a graph G of order n such that $\chi_{ir}(G) = a$ and $\chi_{ir}(\overline{G}) = b$?*

4. Some Classes of Disconnected Graphs

We have seen in our study of irregular chromatic numbers of disconnected graphs whose components are complete multipartite graphs play an important role. In this section, we determine the irregular chromatic numbers of these classes of disconnected graphs.

Theorem 4.1. *Let $G = pK_n$ where $p \geq 2$ and $n \geq 2$. Then*

$$\chi_{ir}(G) = k \text{ if and only if } \binom{k-1}{n} + 1 \leq p \leq \binom{k}{n}.$$

Proof. It is easy to see that the result holds for $n = 2$. For $n \geq 3$, let $G_i \cong K_n$ be a component of G for $1 \leq i \leq p$. Since $\chi_{ir}(K_n) = n$, any irregular coloring of K_n must assign n distinct colors to the n vertices of K_n . First suppose that $\chi_{ir}(G) = k$. We show that $\binom{k-1}{n} + 1 \leq p \leq \binom{k}{n}$. Assume, to the contrary, that $p \geq \binom{k}{n} + 1$ or $p \leq \binom{k-1}{n}$. We consider these two cases.

Case i. $p \geq \binom{k}{n} + 1$.

Let c be an irregular k -coloring of G . Thus c assigns n distinct colors to the n vertices of $G_i \cong K_n$ for $1 \leq i \leq p$. Since there are $\binom{k}{n}$ distinct n -element subsets of the set $\{1, 2, \dots, k\}$ and $p \geq \binom{k}{n} + 1$, it follows that there exist two components G_s and G_t , where $1 \leq s \neq t \leq p$ such that the color sets of G_s and G_t are the same, that is, $\{c(v) : v \in V(G_s)\} = \{c(v) : v \in V(G_t)\}$. Thus there exist $x \in V(G_s)$ and $y \in V(G_t)$ such that $c(x) = c(y)$. However then, $\text{code}_c(x) = \text{code}_c(y)$, which is a contradiction.

Case ii. $p \leq \binom{k-1}{n}$.

Let $A_1, A_2, \dots, A_{\binom{k-1}{n}}$ be the $\binom{k-1}{n}$ distinct n -element subsets of the set $\{1, 2, \dots, k-1\}$. Since $p \leq \binom{k-1}{n}$, we can define a coloring c' of G by assigning the n distinct colors in A_i to the n vertices of $V(G_i)$ for $1 \leq i \leq p$. We show that c' is irregular. For each integer i with $1 \leq i \leq p$, let c_i be the coloring of G_i induced by the coloring c' of G . Then c_i is irregular for $1 \leq i \leq p$. Let $x, y \in V(G)$. If $x, y \in V(G_i)$ for some i with $1 \leq i \leq p$, then $\text{code}_{c'}(x) = \text{code}_{c_i}(x)$ and $\text{code}_{c'}(y) = \text{code}_{c_i}(y)$. Since c_i is irregular, $\text{code}_{c_i}(x) \neq \text{code}_{c_i}(y)$ and so $\text{code}_{c'}(x) \neq \text{code}_{c'}(y)$. Thus, we may assume that $x \in V(G_i)$ and $y \in V(G_j)$ where $1 \leq i, j \leq p$ and $i \neq j$. Since $A_i \neq A_j$, there exists $a \in A_i$ such that $a \notin A_j$. If $c'(x) \neq c'(y)$, then $\text{code}_{c'}(x) \neq \text{code}_{c'}(y)$; while if $c'(x) = c'(y)$, then x is adjacent to a vertex colored a by c' and y is adjacent to no vertex colored a . Thus $\text{code}_{c'}(x) \neq \text{code}_{c'}(y)$. Therefore, c' is an irregular coloring with at most $k-1$ colors. Thus $\chi_{ir}(G) \leq k-1$, which is a contradiction.

For the converse, suppose that $\binom{k-1}{n} + 1 \leq p \leq \binom{k}{n}$ and we show that $\chi_{ir}(G) = k$. Let $B_1, B_2, \dots, B_{\binom{k}{n}}$ be the $\binom{k}{n}$ distinct n -element subsets of the set $\{1, 2, \dots, k\}$. Since $p \leq \binom{k}{n}$, we can define a coloring of G by assigning the n distinct colors in B_i to the n vertices of $V(G_i)$ for $1 \leq i \leq p$. By the argument used in Case 2, this coloring is irregular and uses at most k colors. Thus $\chi_{ir}(G) \leq k$. On the other hand, since $p \geq \binom{k-1}{n} + 1$ and there are $\binom{k-1}{n} + 1$ distinct subsets in $\{1, 2, \dots, k-1\}$, the argument used in Case 1 shows that there is no irregular coloring of G using $k-1$ or fewer colors. Thus $\chi_{ir}(G) \geq k$ and so $\chi_{ir}(G) = k$. \square

By Theorem 4.1, if $G = pK_n$ where $p \geq 2$ and $n \geq 3$ having $\chi_{ir}(G) = k$, then

$$\binom{k}{n}/p \geq 1 \text{ and } \binom{k-1}{n}/p < 1.$$

Thus k is the smallest positive integer such that $\binom{k}{n}/p \geq 1$. The following is a consequence of Proposition 2.4 and Theorem 4.1.

Corollary 4.2. *Let $G = p_1K_{n_1} \cup p_2K_{n_2} \cup \dots \cup p_tK_{n_t}$, where $p_i \geq 1$ and $n_1 < n_2 < \dots < n_p$ for $1 \leq i \leq t$. For each integer i with $1 \leq i \leq t$, let k_i be the smallest positive integer such that $\binom{k_i}{n_i}/p_i \geq 1$. Then*

$$\chi_{ir}(G) = \max\{k_i : 1 \leq i \leq t\}.$$

Proposition 4.3. *Let $p \geq 2$ and $r \geq 2$ be integers. If s is the smallest integer such that $p \leq \frac{1}{2} \binom{2r}{r} \binom{s}{2}$, then $\chi_{ir}(pK_{r,r}) \leq sr$. Furthermore, if $p = \frac{1}{2} \binom{2r}{r} \binom{s}{2}$, then $\chi_{ir}(pK_{r,r}) = sr$.*

Proof. If $p < \frac{1}{2} \binom{2r}{r} \binom{s}{2}$, then $\frac{1}{2} \binom{2r}{r} \binom{s}{2} K_{r,r} = pK_{r,r} \cup p'K_{r,r}$, where $p + p' = \frac{1}{2} \binom{2r}{r} \binom{s}{2}$. It then follows by Theorem 2.3 that $\chi_{ir}(pK_{r,r}) \leq \chi_{ir}(\frac{1}{2} \binom{2r}{r} \binom{s}{2} K_{r,r})$. Thus it suffices to show that

$$\chi_{ir}(\frac{1}{2} \binom{2r}{r} \binom{s}{2} K_{r,r}) \leq sr.$$

Let $G = \frac{1}{2} \binom{2r}{r} \binom{s}{2} K_{r,r}$ and let C_1, C_2, \dots, C_s be pairwise disjoint sets of r colors. We construct an irregular (sr) -coloring of G by the following three steps.

Step 1. We construct an irregular $(2r)$ -coloring of $\frac{1}{2} \binom{2r}{r} K_{r,r}$ using the $2r$ colors in $C_1 \cup C_2$. For each integer i with $1 \leq i \leq \frac{1}{2} \binom{2r}{r}$, let G_i be a copy of $K_{r,r}$ in $\frac{1}{2} \binom{2r}{r} K_{r,r}$ whose partite sets are U_i and W_i . Let

$$A_1, B_1, A_2, B_2, \dots, A_{\frac{1}{2} \binom{2r}{r}}, B_{\frac{1}{2} \binom{2r}{r}}$$

be the $\binom{2r}{r}$ distinct r -element subsets of $C_1 \cup C_2$ such that $A_i \cap B_i = \emptyset$ for $1 \leq i \leq \frac{1}{2} \binom{2r}{r}$. Then the $(2r)$ -coloring c of $\frac{1}{2} \binom{2r}{r} K_{r,r}$ is defined by assigning the distinct r colors in A_i to the r vertices in U_i and assigning the distinct r colors in B_i to the r vertices in W_i for $1 \leq i \leq \frac{1}{2} \binom{2r}{r}$. We show that c is irregular. Let x and y be two vertices of $\frac{1}{2} \binom{2r}{r} K_{r,r}$. If $x, y \in V(G_i)$ for some i with $1 \leq i \leq \frac{1}{2} \binom{2r}{r}$, then $c(x) \neq c(y)$ and so $\text{code}(x) \neq \text{code}(y)$. If $x \in V(G_i)$ and $y \in V(G_j)$, where $i \neq j$, then $\{c(v) : v \in W_i\}$ and $\{c(v) : v \in W_j\}$ are two distinct r -element subsets of $C_1 \cup C_2$. This implies that $\text{code}(x) \neq \text{code}(y)$. Therefore, for each pair i, j with $1 \leq i \neq j \leq s$, we can define an irregular $(2r)$ -coloring of $\frac{1}{2} \binom{2r}{r} K_{r,r}$ using the $2r$ colors in $C_i \cup C_j$.

Step 2. Let a, b, c, d be integers such that $a \neq b$, $c \neq d$, and $\{a, b\} \neq \{c, d\}$. It is possible that $\{a, b\} \cap \{c, d\} \neq \emptyset$. Let F and H be two copies of $\frac{1}{2} \binom{2r}{r} K_{r,r}$, let c_1 be an irregular $(2r)$ -coloring of H using the $2r$ colors in $C_a \cup C_b$, and let c_2 be an irregular $(2r)$ -coloring of F using the $2r$ colors in $C_c \cup C_d$ as defined in Step 1. Let c be a coloring of $H \cup F$ extended from the colorings c_1 and c_2 . We show that c is irregular. Let x and y be two vertices of $H \cup F$. If $x, y \in V(H)$ or $x, y \in V(F)$, say $x, y \in V(H)$, then $\text{code}_c(x) = \text{code}_{c_1}(x)$ and $\text{code}_c(y) = \text{code}_{c_1}(y)$. Since c_1 is an irregular coloring of H , it follows that $\text{code}_c(x) \neq \text{code}_c(y)$. Thus, we may assume that $x \in V(H)$ and $y \in V(F)$. Suppose that x belongs to a component

$H_1 = K_{r,r}$ of H and y belongs to a component $F_1 = K_{r,r}$ of F . If $c(x) \neq c(y)$, then $\text{code}_c(x) \neq \text{code}_c(y)$. Thus we may assume that $c(x) = c(y)$ and so $\{a, b\} \cap \{c, d\} \neq \emptyset$. Assume, without loss of generality, that $a = c$ and so $b \neq d$. Hence the vertices of H are colored with the $2r$ colors in $C_a \cup C_b$ and the vertices of F are colored with the $2r$ colors in $C_a \cup C_d$. Since (1) $|\{c(v) : x \in N(x)\}| = |\{c(v) : x \in N(y)\}| = r$, (2) $c(x) = c(y)$, and (3) the sets $C_a \cup C_b$ and $C_a \cup C_d$ have exactly r colors in common, it follows that $\{c(v) : x \in N(x)\} \neq \{c(v) : x \in N(y)\}$. Therefore, $\text{code}_c(x) \neq \text{code}_c(y)$.

Step 3. Let $G = T_1 \cup T_2 \cup \dots \cup T_{\binom{s}{2}}$, where $T_i = \frac{1}{2} \binom{2r}{r} K_{r,r}$ for $1 \leq i \leq \binom{s}{2}$ and let $S_1, S_2, \dots, S_{\binom{s}{2}}$ be the $\binom{s}{2}$ distinct 2-element subsets of the set $\{1, 2, \dots, s\}$. For each integer i with $1 \leq i \leq \binom{s}{2}$, let $S_i = \{i_1, i_2\}$ and define an irregular $(2r)$ -coloring c_i of T_i using the $2r$ colors in $C_{i_1} \cup C_{i_2}$ as described in Step 1. Now let c be the coloring of G extended from the $\binom{s}{2}$ colorings $c_1, c_2, \dots, c_{\binom{s}{2}}$. It remains to show that c is irregular. Let x and y be two vertices of G . If $x, y \in V(T_i)$ for some integer i with $1 \leq i \leq \binom{s}{2}$, then $\text{code}_c(x) = \text{code}_{c_i}(x)$ and $\text{code}_c(y) = \text{code}_{c_i}(y)$. Since c_i is an irregular coloring of T_i , it follows that $\text{code}_c(x) \neq \text{code}_c(y)$. Thus, we may assume that $x \in V(T_i)$ and $y \in V(T_j)$, where $i \neq j$. Let c_{ij} be the coloring of $T_i \cup T_j$ extended from c_i and c_j . By Step 2, the coloring c_{ij} is irregular and $\text{code}_{c_{ij}}(x) \neq \text{code}_{c_{ij}}(y)$. Since $\text{code}_c(v) = \text{code}_{c_{ij}}(v)$ for all $v \in V(T_i \cup T_j)$, it follows that $\text{code}_c(x) \neq \text{code}_c(y)$. Therefore, c is an irregular coloring of G using sr colors and so $\chi_{ir}(G) \leq sr$. \square

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