

# Graphs of Functions of Two Variables and Contour Diagrams

In the previous section we introduced functions of two variables. We presented those functions primarily as tables. We examined the differences between an equation graphed in 2-space and 3-space. The goal of this section is to introduce a variety of graphs of functions of two variables.

## The Graph of a Function of Two Variables

The graph of a function of two variables,  $f(x, y)$ , is the set of all points  $(x, y, z)$  such that  $z = f(x, y)$ . In general, the graph of a function of two variables is a surface in 3-space.

### Example 1:

Describe and graph the function  $f(x, y) = xy$ .

### Solution:

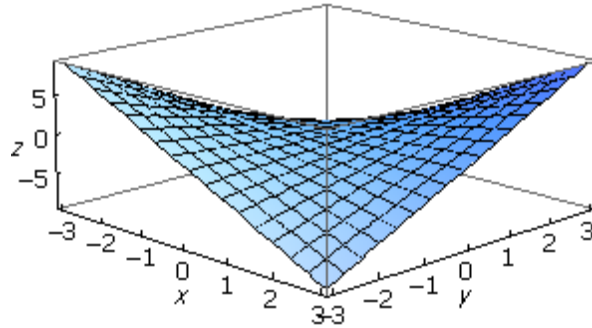
We begin by making a table of values. Then we attempt to plot those points in 3-space and connect the points. As we include more and more points, our graph becomes the desired surface.

		y						
		-3	-2	-1	0	1	2	3
x	-3	9	6	3	0	-3	-6	-9
	-2	6	4	2	0	-2	-4	-6
	-1	3	2	1	0	-1	-2	-3
	0	0	0	0	0	0	0	0
	1	-3	-2	-1	0	1	2	3
	2	-6	-4	-2	0	2	4	6
	3	-9	-6	-3	0	3	6	9

**Table 1: Values of  $f(x, y) = xy$**

We see that if  $x$  or  $y$  is 0, the function is 0. As  $x$  and  $y$  both increase, the values of the function get larger. For a fixed value of either  $x$  or  $y$ , the function looks like a line.

Using this information, we can attempt to make a sketch of the function.



**Figure 1: Graph of  $f(x, y) = xy$**

We remark that as  $x$  and  $y$  are both positive or both negative, the function values are getting larger and positive. When one is positive and the other is negative, the function values are getting larger and negative. This is illustrated by the slants of the surface.

**Example 2:**

Graph the function  $g(x, y) = x^2 + y^2$ .

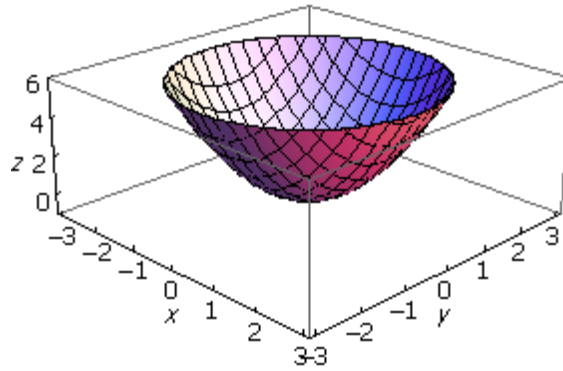
**Solution:**

Again, we can start by creating a table of values.

		y						
		-3	-2	-1	0	1	2	3
x	-3	18	13	10	9	10	13	18
	-2	13	8	5	4	5	8	13
	-1	10	5	2	1	2	5	10
	0	9	4	1	0	1	4	9
	1	10	5	2	1	2	5	10
	2	13	8	5	4	5	8	13
	3	18	13	10	9	10	13	18

**Table 2: Values of  $g(x, y) = x^2 + y^2$**

Notice that there is a lot of symmetry in the table. For any fixed value of  $x$ , or any fixed value of  $y$ , the values increase as we move further away from 0. It has a minimum value of 0 which occurs at  $x = 0, y = 0$ . Also, swapping  $x$  and  $y$  results in the same value. A graph of the function appears below.



**Figure 2: Graph of  $g(x, y) = x^2 + y^2$**

When we considered functions and graphs of one variable, one of the first things we did was to transform those graphs through shifts and stretches. We can do the same thing with functions of two variables.

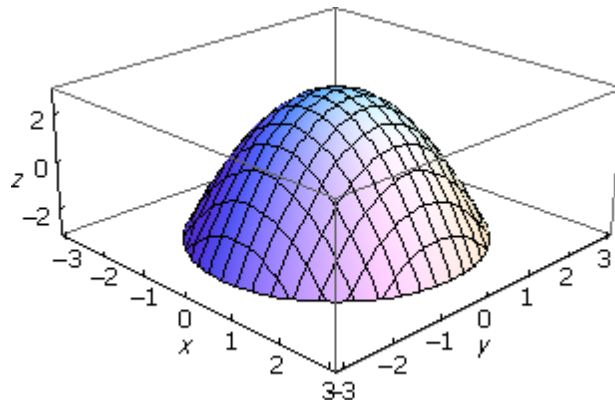
**Example 3:**

Using the function from Example 2, describe and graph the following functions:

- (i)  $f(x, y) = 3 - x^2 - y^2$
- (ii)  $g(x, y) = (x - 1)^2 + (y + 1)^2$
- (iii)  $h(x, y) = 4x^2 + y^2$

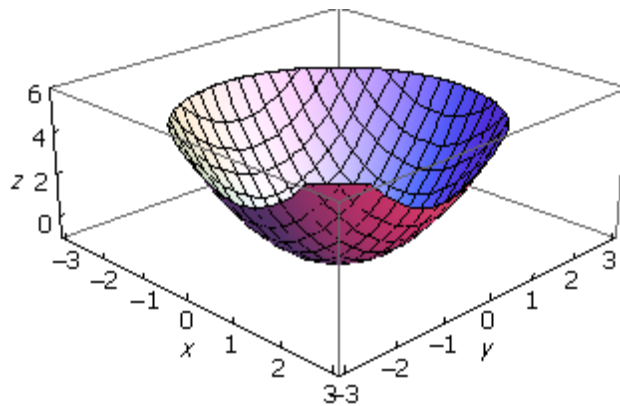
**Solution:**

- (i)  $f(x, y)$  looks like  $x^2 + y^2$ , except it has been shifted up by 3 units and opens downwards instead of upwards. It has a maximum value at  $(0, 0, 3)$ .



**Figure 3: Plot of  $3 - x^2 - y^2$**

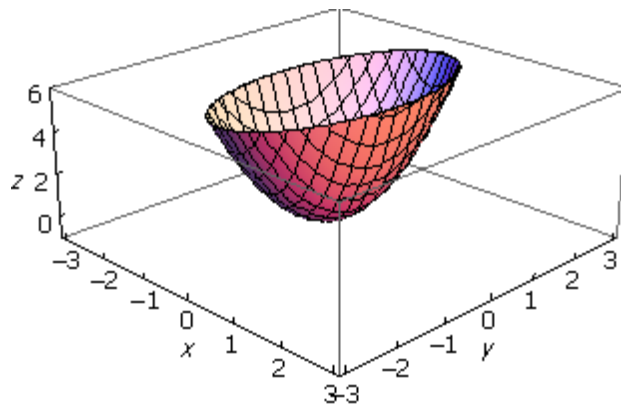
- (ii)  $g(x, y)$  looks like  $x^2 + y^2$ , except it has been translated by 1 unit to the right in the  $x$ -direction and 1 unit to the left in the  $y$ -direction. It has its minimum value at the point  $(1, -1, 0)$ .



**Figure 4: Plot of  $(x - 1)^2 + (y + 1)^2$**

It is a little difficult to see that this is the same shape as  $x^2 + y^2$ . The reason for the slight difference is the clipping introduced with the window size. Compared the Figure 2, the minimum point does appear to be at the point  $(0, -1, 0)$ .

- (iii)  $h(x, y)$  looks like  $x^2 + y^2$ , except in the  $x$ -direction, the values have been compressed (which has the effect of making the  $y$ -values appear to be stretched). As compared to the circular bowl of the graph of  $x^2 + y^2$ ,  $h(x, y)$  will look like an elliptical bowl.



**Figure 5: Plot of  $4x^2 + y^2$**

A good way to visualize what a graph looks like in 3-space is to consider **cross-sections** of the function. That is, we fix either  $x$  or  $y$  and see what the resulting graph looks like. This is like standing on one of the axes and describing what we see. We can also fix the  $z$ -value. The resulting curve is called a **level curve**. The collection of such curves is called a **contour diagram**. The more closely spaced the level curves, the more the function is changing. The more widely spaced the level curves, the function is more constant. Taking the cross-sections and contour diagram into account, we can reconstruct what the shape looks like.

### Cross-Sections and Contour Diagram of a Graph

For a function  $z = f(x, y)$ , the function we get by holding  $x$  fixed and letting  $y$  vary is called a cross-section of  $f(x, y)$ , with  $x$  fixed.

The graph of the cross-section of  $f(x, y)$  with  $x = c$  is the curve we get by intersecting the graph of  $f(x, y)$  with the plane  $x = c$ . Similarly, if we fix  $y$  and let  $x$  vary, we get the cross-section of  $f(x, y)$  with  $y$  fixed.

If we fix the  $z$ -value in the graph, the resulting curve is called a level curve and the collection of these curves is called the contour diagram of the function  $f(x, y)$ .

#### Example 4:

What shapes are the cross-sections of the function  $z = f(x, y) = x^2 + y^2$ ? What about the contour diagram of  $f(x, y)$ ?

#### Solution:

The cross-sections of  $x^2 + y^2$  look like upwards pointing parabolas since setting  $y = c$ , results in a curve which looks like  $z = x^2 + c^2$ , and if we fix  $x = c$ , then the resulting curve is  $z = c^2 + y^2$ . Both of these are upwards pointing parabolas with an intercept of  $c^2$ .

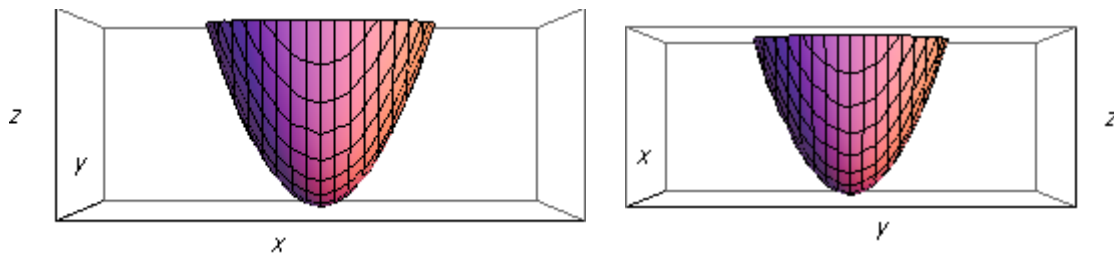


Figure 6: Cross-sections of  $x^2 + y^2$

The level curves of the function are circles, since  $k = x^2 + y^2$  denotes a circle with radius  $\sqrt{k}$ . Collectively, we have the following contour diagram of the function.

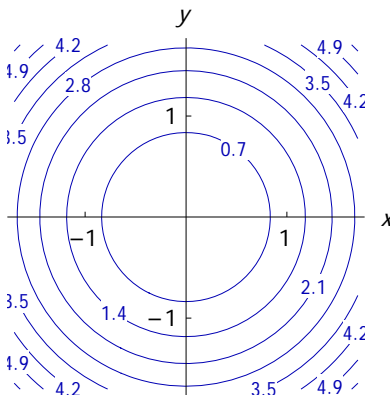


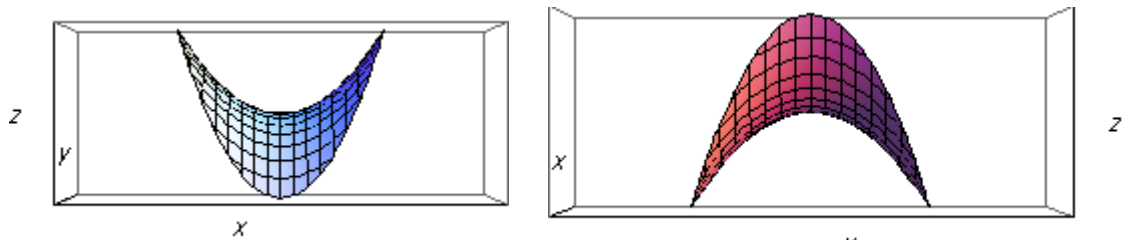
Figure 7: Contour Diagram of  $x^2 + y^2$

**Example 5:**

By first considering cross-sections and the contour diagram, sketch the graph of the function  $g(x, y) = x^2 - y^2$ .

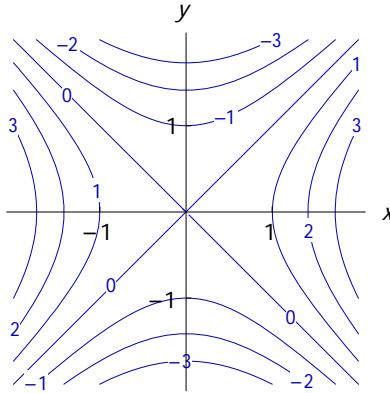
**Solution:**

Similar to Example 4, we begin with cross-sections. Notice that if we set  $y = c$ , we end up with  $z = x^2 - c^2$ , which is an upwards pointing parabola. But if we  $x = c$ , we end up with  $z = c^2 - y^2$ , which is a downwards pointing parabola.



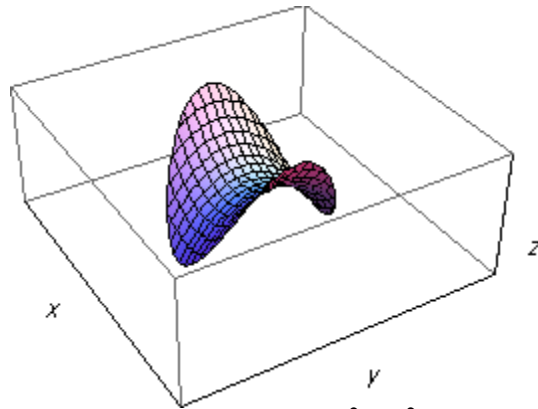
**Figure 8: Cross-sections of  $x^2 - y^2$**

A contour diagram shows hyperbolas (as compared to circles). Again, this makes sense because if we set  $z = k$ , we have  $k = x^2 - y^2$ , which is the equation of a hyperbola.



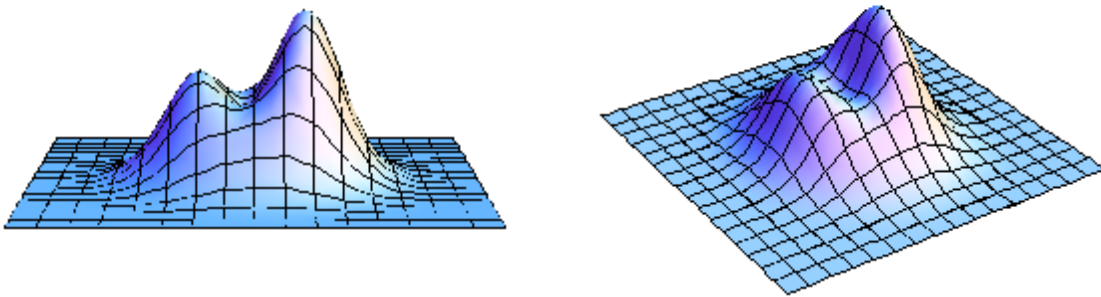
**Figure 9: Contour Diagram of  $x^2 - y^2$**

Putting this all together, we get an idea of what the resulting shape looks like. From the  $y$ -axis, it looks like an upwards pointing parabola, but from the  $x$ -axis it looks like a downwards pointing parabola. The best way to describe this shape is what a horse saddle looks like or what a Pringles potato chip looks like.



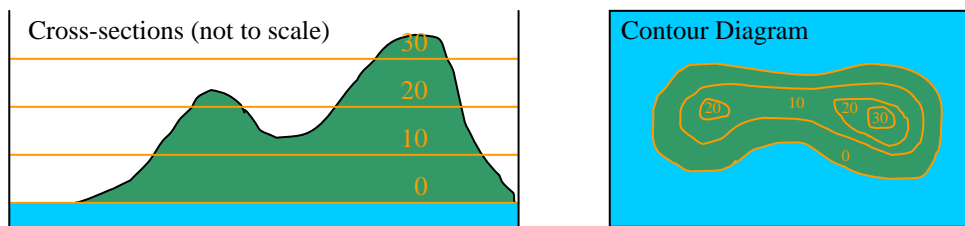
**Figure 10: The graph of  $x^2 - y^2$  (a saddle)**

Contour diagrams are the preferred manner to present a topographic map. Consider the following island, show below with two different views.



**Figure 11: An Island**

We present both a cross-section of the island and a contour diagram.

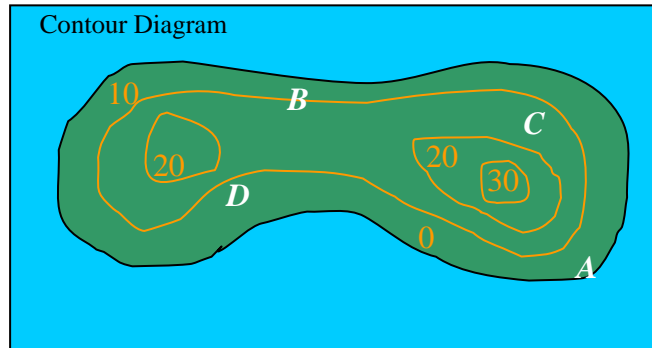


**Figure 12: Cross-sections and Contour Diagram of an Island**

Each orange curve in the contour diagram corresponds to a horizontal slice of the island. We see that the perimeter of the island is 0 feet above sea level. We also see that the level curves are close near the peak on the right, suggesting a fairly steep mountain.

The majority of the island is about 10 feet above sea level. We notice that there are two peaks in the contour diagram. The left one corresponds to a small hill that is slightly more than 20 feet above sea level. The right one corresponds to one that is slightly more than 30 feet above sea level. (This corresponds to the cross-sections we see on the left.)

A common use for a topographic map is to determine the elevation at a specified location. The map below is an enlargement of the map of the island from above. Each of the letters from *A* to *D* represent locations for which we wish to determine elevation.



**Figure 13: Contour Diagram of an Island with Points of Interest**

**Example 6:**

Use the map to estimate the elevation of each of the 4 points. (Assume elevations are given in feet).

**Solution:**

Point *A* sits right on the 0 ft contour line. Since all points on this line have an elevation of 0 ft, the elevation of point *A* is zero.

Point *B* sits right on the 10 ft contour line. Since all points on this line have an elevation of 10 ft, the elevation of point *B* is 10 ft.

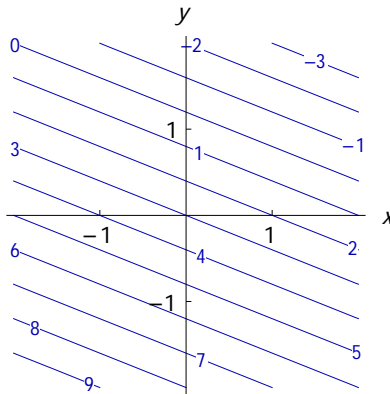
Point *C* does not sit directly on a contour line so we can not determine the elevation precisely. We do know that point *C* is between the 10ft and 20 ft contour lines so its elevation must be greater than 10 ft and less than 20 ft. Because point *C* is halfway between these contour lines we can estimate the elevation is about 15 feet (This assumes that the slope is constant between the two contour lines, this may not be the case.)

Just as with point *C* above, we need to estimate the elevation of point *D* somewhere between the 0 ft and 10 ft contour lines it lies in between. Because this point is closer to the 10 ft line than the 0 ft line we estimate an elevation closer to 10. In this case 8 ft seems reasonable. Again this estimation makes the assumption of a constant slope between these two contour lines.



**Example 7:**

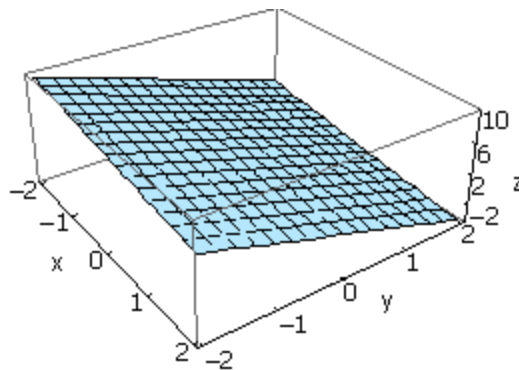
Given the following contour diagram for  $f(x, y)$ , sketch the graph of the function.



**Figure 14: Contour diagram of  $f(x, y)$**

**Solution:**

We notice that as we move from the bottom-left to the top-right, the function is decreasing in value. The spacing between level curves suggests that the decrease is constant. It is almost as if we are looking at a sheet of paper that is declining. In fact, that is the shape that we are dealing with.



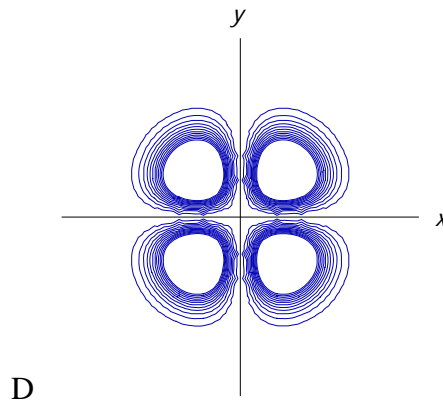
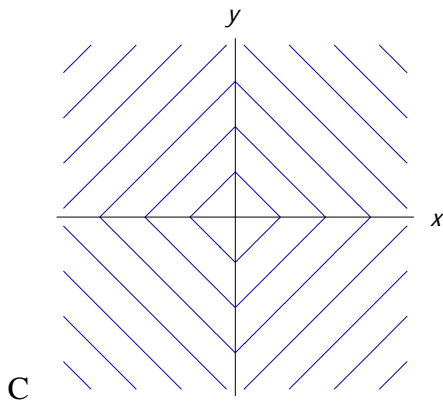
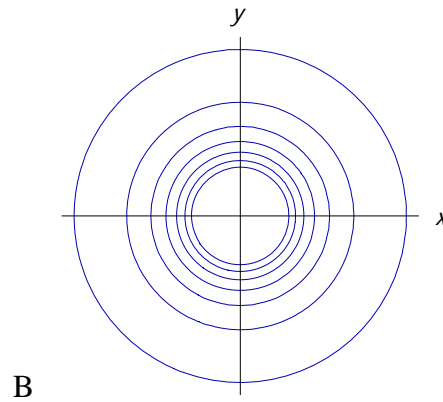
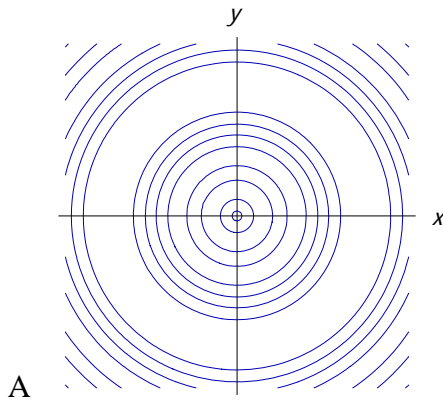
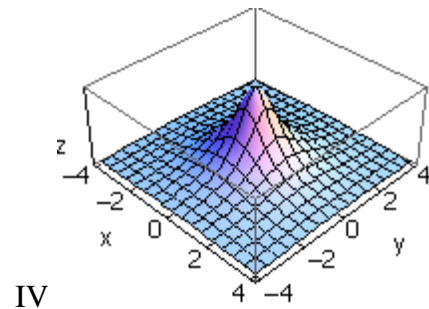
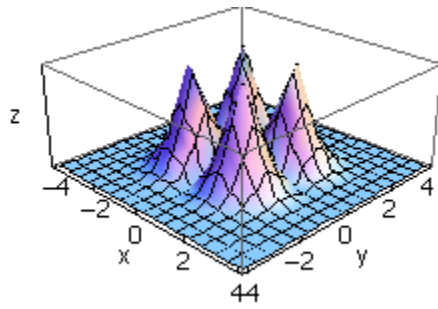
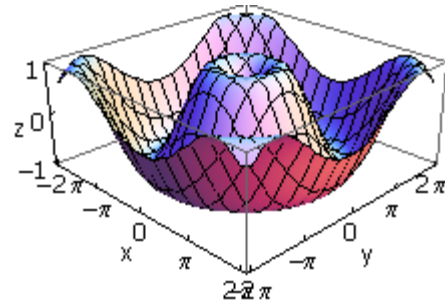
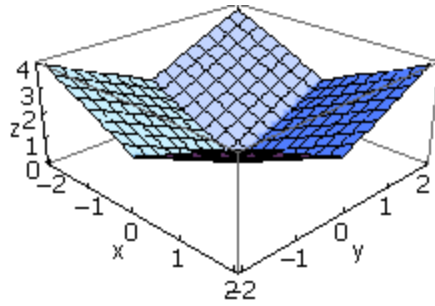
**Figure 15: The graph of  $f(x, y)$**

As we shall see in the next section, whenever the level curves are evenly spaced lines, the resulting shape will always be the equation of a plane. We will also discuss ways to determine the equation of the plane using the contour plot in more detail.

For the curious reader, the equation of the above graph is given by  $3 - x - (5/2)y$ .

**Example 8:**

Match the following surfaces with their contour diagrams.



**Solution:**

First, we identify that graph I corresponds to the contour diagram C. This is because of the symmetry in all directions and the fact that the function is constant for diagonal lines.

Next, we see that graph III corresponds to the contour diagram D. We notice that graph III has four peaks that appear to be symmetric. The contour diagram D exhibits this same behavior.

This leaves graphs II and IV and contour diagrams A and B. We claim that graph IV corresponds to contour diagram B. The tightly packed lines on contour diagram B suggest that the graph is changing rapidly near the origin and slowly further away. Also, we notice that the graph exhibits circular symmetry about the origin, which means that as we spin around the origin, the graph looks the same. This is what graph IV shows.

Thus, we are left claiming that graph II corresponds to contour diagram A. Let us verify this fact. Again, we notice circular symmetry about the origin. In contrast with contour diagram B, though, we notice that there is a place a little away from the origin where the graph appears to not change by much, and then it starts changing quite a bit again. The fairly constant place is roughly where the graph is obtaining a minimum (or maximum) value. Looking at graph II, we see that there is such a place on the graph.