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An example and transition function equicontinuity

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Abstract

Conditions are given for a Markov chain to be extendable to the one point compactification of the integers to yield a transition function taking continuous functions into continuous functions. These chains give examples of equicontinuous and nonequicontinuous transition operators.

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0. Introduction

Let T be the transition operator of a Markov process with state space $\mathbb{X}$. If the state space is a compact topological space and the T maps continuous functions into continuous functions, the Markov process is often referred to as a Feller process

$$Tf(x) = \int t(x, \mathrm{d}y)f(y).$$

The transition operator is called (quasi) equicontinuous if the sequence of continuous functions $\{T^n f,\ n = 1, 2, \ldots\}$ is (quasi) equicontinuous for each continuous f (see Rosenblatt, 1964). The sequence $\{T^n f\}$ is said to be *equicontinuous* if for each $\varepsilon > 0$ and each $x \in \mathbb{X}$ there is a neighborhood $N(x)$ such that

$$\sup_n \sup_{y \in N(x)} |T^n f(x) - T^n(y)| < \varepsilon.$$

The sequence $\{T^n\}$ is said to be *quasi-equicontinuous* if for any sequence $x_\alpha \to x(\alpha \to \infty)$ given any $\varepsilon > 0$ and α_0 there is a finite set of indices $\alpha_i \geqslant \alpha_0$, $i = 1, \ldots, s$, such that

$$\sup_n \min_{1 \leqslant i \leqslant s} |T^n f(x_{\alpha_i}) - T^n f(x)| \leqslant \varepsilon$$

(See Dunford and Schwartz, 1958, pp. 266–269).

Conditions like these are often useful in describing the limiting behavior of the sequence of operators T^n as $n \to \infty$ (see De Leeuw and Glicksberg, 1961).

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One of the objects of this paper is to give a simple set of examples that illustrate how (quasi) equicontinuity of an operator T may or may not hold. We start by considering a countable state Markov chain with transition matrix $P = (P_{i,j},\ i,j = 1, 2, \ldots)$

$$P_{i,j} = P(\mathbb{X}_{n+1} = j | \mathbb{X}_n = i).$$

The chain has the discrete state space $I = \{x = i,\ i = 1, 2, \ldots\}$ with the discrete topology in which every one point set is open. In the one point compactification I^* of I (with the additional point ∞) the open sets of I^* are the open sets of I and the complements in I^* of the closed compacts of I (the finite sets of I). Our first object is to determine the extensions of the Markov chain with transition matrix P to I^* which are Feller transition operators.

Lemma 0.1. *There is an extension of P to I^* that is a Feller transition function if and only if*

$$\lim_{i\to\infty} P_{i,j}, \quad j = 1, 2, \ldots \tag{1}$$

exist. In the extension

$$P_{\infty,j} = \lim_{i\to\infty} P_{i,j}, \quad j = 1, 2, \ldots$$

and

$$P_{\infty,\infty} = 1 - \sum_{j<\infty} P_{\infty,j}.$$

Consider a Markov chain with transition probabilities

$$P_{i,j} = \begin{Bmatrix} \mu_i & \text{if } j = i+1 \\ 1-\mu_i & \text{if } j = 1 \\ 0 & \text{if } j \neq 1, i+1 \end{Bmatrix}. \tag{2}$$

Notice that

$$\lim_{i\to\infty} P_{i,j} = 0 \quad \text{if } j \neq 1$$

so that the chain satisfies condition (1) if

$$\mu = \lim_{j\to\infty} \mu_j \tag{3}$$

exists. The chains of this type satisfying (2), (3) are all familiar and will be called chains of type A.

Proposition 0.2. *The extension of positive recurrent chains of type A to I^* is (quasi) equicontinuous if and only if*

$$\mu = \lim \mu_j < 1.$$

Proof of Lemma 0.1. The function that is one on j ($j = 1, 2, \ldots$) and zero elsewhere on I as extended to I^* must be zero at ∞ to be continuous. The extended operator (of P) acting on the function is the function $P_{i,j},\ i = 1, 2, \ldots$. For the extension of this to be continuous we must have

$$\lim_{i\to\infty} P_{i,j} \tag{4}$$

exist and set the value at ∞ equal to (4). So limit (4) must exist for $j = 1, 2, \ldots$ confirming (1). The operator P extended to I^* must clearly have

$$P_{i,\infty} = 0 \quad \text{for } i = 1, 2, \ldots$$

$$P_{\infty,j} = \lim_{i\to\infty} P_{i,j} \quad \text{for } j = 1, 2, \ldots$$

and

$$P_{\infty,\infty} = 1 - \sum_{j<\infty} P_{\infty,j}.$$

Let us now show that the extended operator on I^* takes continuous f into a continuous function on I^*. Let $f=(f_j)$ be continuous function on I^*. Clearly

$$f_\infty = \lim_{j\to\infty} f_j.$$

Now for $i<\infty$

$$\begin{aligned}\sum_j P_{i,j} f_j &= f_\infty + \sum_j P_{i,j}(f_j - f_\infty)\\ &= f_\infty + \sum_j P_{\infty,j}(f_j - f_\infty) + \sum_j (P_{i,j} - P_{\infty,j})(f_j - f_\infty)\end{aligned} \tag{5}$$

while

$$\sum_j P_{\infty,j} f_j = P_{\infty,\infty} f_\infty + \sum_{j<\infty} P_{\infty,j} f_j = f_\infty + \sum_{j<\infty} P_{\infty,j}(f_j - f_\infty). \tag{6}$$

Clearly $\sum_j |P_{i,j} - P_{\infty,j}| \leqslant 2$.

For any $\varepsilon > 0$ there is an $N(\varepsilon)$ such that

$$|f_j - f_\infty| < \varepsilon \quad \text{for } j > N(\varepsilon).$$

However, there is an $M(\varepsilon)$ such that for $i > M(\varepsilon)$

$$\sum_{j=1}^{N(\varepsilon)} |P_{i,j} - P_{\infty,j}| < \varepsilon.$$

Therefore for $i > M(\varepsilon)$

$$\sum_j |P_{i,j} - P_{\infty,j}||f_j - f_\infty| \leqslant 2\varepsilon(1 + \sup_j |f_j|).$$

Since (5) tends to (6) as $i \to \infty$ the extended operator on I^* is Feller. □

Proof of Proposition 0.2. Let us first note that

$$P_{j,s}^{(n)} = \sum_{m=1}^{n} f_{j,s}^{(m)} P_{s,s}^{(n-m)}, \tag{7}$$

where $f_{j,s}^{(m)}$ is the probability of first passage from j to s in exactly m steps. If the chain is positive recurrent

$$\sum_{m=1}^{\infty} f_{j,s}^{(m)} = 1 \tag{8}$$

and

$$P_{s,s}^{(n)} \to \frac{1}{m_s}$$

as $n \to \infty$ where m_s is the mean recurrence time of state s. Also

$$f_{j,1}^{(n)} = \mu_j \mu_{j+1} \cdots \mu_{j+n-2}(1 - \mu_{j+n-1})$$

so that

$$\sum_{m=1}^{n} f_{j1}^{(m)} = 1 - \mu_j \mu_{j+1} \cdots \mu_{j+n-1}. \tag{9}$$

Further

$$f_{j,s}^{(n)} = f_{j,1}^{(n-s+1)} \left(\prod_{k=1}^{s-1} \mu_k \right),$$

$$P_{j,s}^{(n)} = P_{j,1}^{(n-s+1)}\left(\prod_{k=1}^{s-1}\mu_k\right). \tag{10}$$

First assume that $\lim_{s\to\infty}\mu_s = 1$. Consider any integer $n > 1$. There is then an integer j_n such that

$$\mu_{j_n+s} > 1 - \frac{1}{n}, \quad s = 1, \ldots, n.$$

Clearly as $n \to \infty$, $j_n \to \infty$. Now

$$\prod_{s=1}^{n}\mu_{j_n+s} \geqslant e^{-1}.$$

For fixed j, $P_{j,1}^{(n)} \to 1/m_1$, as $n \to \infty$ where m_1 is the mean recurrence time for state 1. However,

$$\sum_{m=1}^{n} f_{j_{n,1}}^{(m)} \leqslant 1 - e^{-1}$$

and so

$$P_{j_n,1}^{(n)} \leqslant \frac{(1-e^{-1})}{m_1}.$$

Since there is an oscillation greater than or equal to e^{-1}/m_1 in every neighborhood of ∞, the Markov chain is not equicontinuous.

Now assume

$$\lim_{s\to\infty}\mu_s < 1.$$

There is then an integer m such that

$$\mu_s \leqslant \beta < 1 \quad \text{for } s > m. \tag{11}$$

Using (7)–(11), it is clear that for each s

$$P_{j,s}^{(m)} \to \frac{1}{m_1}\left(\prod_{k=1}^{s-1}\mu_k\right) = \frac{1}{m_s}$$

uniformly in j as $n \to \infty$. Also

$$\sum_{\infty > k \geqslant r} P_{j,k}^{(n)} = \left(1 - \sum_{s=1}^{r-1} P_{j,s}^{(n)}\right) \to 1 - \sum_{s=1}^{r-1}\frac{1}{m_s}$$

for each r converges uniformly in j. All this is enough to show that in this case the Markov chain process is equicontinuous. □

The results used on Markov chains can all be found in Feller (1968). Notice that if $\lim_{s\to\infty}\mu_s < 1$ there is a unique invariant probability distribution for the transition function on I^*. If $\lim_{s\to\infty}\mu_s = 1$ there are two mutually singular invariant probability distribution for the transition function on I^*, one of which has mass 1 at ∞. However, its support is in the closure of the support of the other invariant probability distribution even though the two measures are mutually singular.

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