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Line spectral analysis for harmonizable processes

(nonstationarity/spectral estimation/asymptotic bias and covariance)

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ABSTRACT **Harmonizable processes with spectral mass concentrated on a number of straight lines are considered. The asymptotic behavior of the bias and covariance of a number of spectral estimates is described. The results generalize those obtained for periodic and almost periodic processes.**

Let $\{X_t\}$ be a continuous time parameter harmonizable process continuous in mean square, $-\infty < t < \infty$, with $EX_t \equiv 0$. By this we mean that the covariance function $r(t, \tau) = E(X_t\overline{X}_\tau)$ has a Fourier representation

$$r(t, \tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{it\lambda - i\tau\mu} dF(\lambda, \mu) \quad \textbf{[1]}$$

with $F(\lambda, \mu)$ a function of bounded variation. This implies that X_t itself has a Fourier representation in mean square

$$X_t = \int_{-\infty}^{\infty} e^{it\lambda} dZ(\lambda) \quad \textbf{[2]}$$

in terms of a random function $Z(\lambda)$ with

$$E(dZ(\lambda)\overline{dZ(\mu)}) = dF(\lambda, \mu). \quad \textbf{[3]}$$

If the process X_t is real-valued,

$$dF(u, v) = dF(-v, -u) = \overline{dF(v, u)} = \overline{dF(-u, -v)}. \quad \textbf{[4]}$$

In the case of a weakly stationary process $r(t, \tau) = r(t - \tau, 0)$ and all the spectral mass is located on the diagonal line $\lambda = \mu$. If the process is periodic with

$$r(t, \tau) = r(t + a, \tau + a)$$

for some period a or almost periodic the spectral mass is located on a finite or countable number of lines in the (λ, μ) plane with slope one. If the process is discretely observed X_n, $n = 0, \pm 1, \pm 2, \ldots$ there is an analogous representation

$$X_n = \int_{-\pi}^{\pi} e^{in\lambda} dW(\lambda), \quad \textbf{[5]}$$

$$dW(\lambda) = \sum_k dZ(\lambda + 2k\pi),$$

with

$$r(n, m) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{in\lambda - im\mu} dH(\lambda, \mu)$$

$$E(dW(\lambda)\overline{dW(\mu)}) = dH(\lambda, \mu) \quad \textbf{[6]}$$

$$dH(\lambda, \mu) = \sum_{j,k} dF(\lambda + 2\pi j, \mu + 2\pi k).$$

The folding of $F(\lambda, \mu)$ to obtain $H(\lambda, \mu)$ is referred to as aliasing.

There is an enormous literature concerned with spectral estimation in the case of stationary processes (1). Recently efforts have been made to obtain analogous results on spectral estimation for periodic and almost periodic processes (2–7). It is well known that one generally does not have consistent estimates of spectral mass for a harmonizable process when the function F (or H) is absolutely continuous with a spectral density function f, $dF(\lambda, \mu) = f(\lambda, \mu)d\lambda d\mu$, with $f(\lambda, \mu) \neq 0$ on a set of positive two-dimensional Lebesgue measure, and one is sampling from the process $X_{-n}, \ldots, X_n$ and $n \to \infty$. A simple example is given by X_0 normal with mean zero and variance one and $X_k \equiv 0$ with probability one for $k \neq 0$. It is clear that consistency of spectral estimates in the case of stationary and periodic processes is due to the fact that the spectral mass is concentrated on lines that in these cases happen to be of slope one. We shall consider spectral estimation for harmonizable processes when the spectrum is concentrated on a finite (or possibly countable) number of lines. For convenience the slope of the lines will be assumed to be positive, though the modification for negative slopes is clear. A simple example of a harmonizable process with spectral mass on lines is given by

$$X_t = Y_t + \sum_{s=1}^{k} \beta_s Y_{\alpha_s t}, \quad \textbf{[7]}$$

where Y_t is stationary and β_s and α_s are real and positive numbers, respectively. The object is to give some insight into an interesting class of nonstationary processes.

Assume that X_t is a harmonizable real-valued process as already specified with

A1. All its spectral mass on a finite number of lines with positive slope

$$u = a_i v + b_i,\ i = 1, \ldots, k.$$

A2. The spectral mass on the line $u = a_i v + b_i$ is given by a continuously differentiable spectral density $f_{a_i,b_i}(v)$ if $a_i \leq 1$. Notice that the real-valued property implies that if $u = av + b$ is a line of spectral mass, then so are the lines $u = av - b$ and $u = a^{-1}v \pm a^{-1}b$. If there are lines of nonzero spectral mass, the diagonal $\lambda = \mu$ must be one of them with positive spectral mass. The condition $dF(u, v) =$

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$\overline{dF(v, u)}$ implies if $u = av + b$ is a line of nonzero spectral mass, then so is $u = a^{-1}(v - b)$ with

$$\overline{f_{a^{-1},-a^{-1}b}}(av + b) = a^{-1}f_{a,b}(v).$$

A3. The spectral densities $f_{a_i,b_i}(v)$ and their derivatives are bounded in absolute value by a function $h(v)$ that is a monotonic decreasing function of $|v|$ that decreases to zero as $|v| \to \infty$ and that is integrable as a function of v.

As already remarked, aliasing or folding of the spectral mass occurs when the process is discretely sampled at times $n = \ldots, -1, 0, 1, \ldots$ rather than continuously. The following simple remark indicates how a process with line spectra may differ from a stationary or almost periodic process in terms of aliasing. The aliasing in the case of a harmonizable process has a more complicated character.

PROPOSITION 1. *Let X_t be a continuous time parameter process continuous in mean square satisfying conditions* A1–A3. *Assume that the lines of spectral support have spectral density nonzero at all points v, $|v| > s$ for some $s > 0$. The process discretely observed X_n then has a countably dense set of lines of support in $[-\pi, \pi]^2$ if and only if one of the lines of spectral support of X_t has irrational slope a.*

The Periodogram

We shall consider spectral estimation for the discretely observed process X_n. The estimates will be obtained by smoothing a version of the periodogram. Before dealing with the spectral estimates, approximations for the mean and covariances of the periodogram are obtained. Let

$$F_n(\lambda) = \sum_{t=-\frac{n}{2}}^{n/2} x_t e^{-it\lambda} \qquad [8]$$

be the finite Fourier transform of the data $x_{-n/2}, \ldots, x_{n/2}$. The periodogram

$$I_n(\lambda, \mu) = \frac{1}{2\pi(n+1)} F_n(\lambda)\overline{F_n(\mu)}, \qquad [9]$$

and it is to be understood that $|\lambda|, |\mu| \le \pi$ with $-\pi$ identified with π so that in effect one is dealing with the torus in (λ, μ). Set

$$D_n(x) = \sum_{t=-\frac{n}{2}}^{n/2} e^{itx} = \frac{\sin\left(\frac{1}{2}(n+1)x\right)}{\sin\frac{1}{2}x},$$

$$\tilde{D}_n(x) = \frac{\sin\left(\frac{1}{2}(n+1)x\right)}{\frac{1}{2}x}.$$

These expressions are versions of the Dirichlet kernel adapted to $(-\pi, \pi]$ and $(-\infty, \infty)$. The following result is useful in obtaining the expressions for the mean and covariance of the periodogram.

LEMMA 1. *If $a > 0$, $|y| < \pi$,*

$$\int_{-\pi}^{\pi} D_n(ax+y)D_n(x)dx = \int_{-\pi}^{\pi} \tilde{D}_n(ax+y)\tilde{D}_n(x)dx + O(\log n) \qquad [10]$$

and

$$\frac{1}{2(n+1)}\int_{-\pi}^{\pi} \tilde{D}_n(ax+y)\tilde{D}_n(x)dx$$

$$= \int_{-\infty}^{\infty} \frac{\sin\left(au + \frac{1}{2}(n+1)y\right)}{\left(au + \frac{1}{2}(n+1)y\right)} \frac{\sin u}{u} du + O\left(\frac{\log n}{n}\right)$$

$$= \pi \frac{\sin\left(\frac{1}{2}\frac{(n+1)y}{a}\min(a,1)\right)}{a\frac{1}{2}(n+1)y/a} + O\left(\frac{\log n}{n}\right) \qquad [11]$$

if $|y| \le a/3$, while if $|y| \ge a/3$ the expression is itself of order log n/n.

Whenever we refer to an expression $\omega = z \bmod 2\pi$ it is understood that $-\pi < \omega \le \pi$. Let $\{u\}$ be the integer ℓ such that $-1/2 < u - \ell \le 1/2$. Our version of $z \bmod 2\pi$ is then $z \bmod 2\pi = z - \{z/(2\pi)\}2\pi$. In the following an approximation is given for $EI_n(\alpha\mu + \omega, \mu)$ with the condition imposed that $\alpha > 0$ and $-\pi < \alpha\mu + \omega, \mu \le \pi$. Let $y = y(k, a, b) = (2\pi ka + (a - \alpha)\mu + b - \omega) \bmod 2\pi$.

THEOREM 1. *The mean*

$$EI_n(\alpha\mu + \omega, \mu) = \sum_{|y(k,a,b)| \le a/3} \sum_{a,b} f_{a,b}(\mu + 2\pi k) \times \frac{\sin\left(\frac{1}{2}\frac{(n+1)y}{a}\min(a,1)\right)}{a\frac{1}{2}(n+1)y/a} + O\left(\frac{\log n}{n}\right), \qquad [12]$$

where in the sum it is understood that the k are integers and the pairs (a, b) correspond to the lines $u = av + b$ in the spectrum of the continuous time parameter process X_t that one is observing at integer t.

In the following result an approximation is given for the $\mathrm{cov}(I_n(\alpha\mu + \omega, \mu), I_n(\alpha'\mu' + \omega', \mu'))$. Here

$$y(1) = (2\pi ka + a\lambda' + b - \lambda) \bmod 2\pi$$

$$y(2) = (2\pi ka' - a'\mu' + \mu + b') \bmod 2\pi$$

$$y(3) = (2\pi ka - a\mu' + b - \lambda) \bmod 2\pi$$

$$y(4) = (2\pi ka' + a'\lambda' + \mu + b') \bmod 2\pi \qquad [13]$$

with $\lambda = \alpha\mu + \omega$, $\lambda' = \alpha'\mu' + \omega'$, $-\pi < \lambda, \lambda', \mu, \mu' \le \pi$. Also (a, b), (a', b') correspond to lines in the spectrum of the continuous time parameter process X_t that one is observing at integer times.

THEOREM 2. *The covariance in the case of a normal process* X_t *is*

$$\operatorname{cov}(I_n(\alpha\mu+\omega,\mu), I_n(\alpha'\mu'+\omega',\mu'))$$

$$= \left[\sum_{\substack{|y(1)|\\ \le \frac{1}{3}a}} \sum_{a,b} f_{a,b}(\lambda'+2\pi k) \times \frac{\sin\left(\frac{1}{2}(n+1)\frac{y(1)}{a}\min(a,1)\right)}{a\,\frac{1}{2}(n+1)\frac{y(1)}{a}} + O\left(\frac{\log n}{n}\right)\right]$$

$$\times \left[\sum_{\substack{|y(2)|\\ \le \frac{1}{3}a'}} \sum_{a',b'} f_{a',b'}(-\mu'+2\pi k) \times \frac{\sin\left(\frac{1}{2}(n+1)\frac{y(2)}{a'}\min(a',1)\right)}{a'\,\frac{1}{2}(n+1)\frac{y(2)}{a'}} + O\left(\frac{\log n}{n}\right)\right]$$

$$+ \left[\sum_{\substack{|y(3)|\\ \le \frac{1}{3}a}} \sum_{a,b} f_{a,b}(-\mu'+2\pi k) \times \frac{\sin\left(\frac{1}{2}(n+1)\frac{y(3)}{a}\min(a,1)\right)}{a\,\frac{1}{2}(n+1)\frac{y(3)}{a}} + O\left(\frac{\log n}{n}\right)\right]$$

$$\times \left[\sum_{\substack{|y(4)|\\ \le \frac{1}{3}a'}} \sum_{a',b'} f_{a',b'}(\lambda'+2\pi k) \times \frac{\sin\left(\frac{1}{2}(n+1)\frac{y(4)}{a'}\min(a',1)\right)}{a'\,\frac{1}{2}(n+1)\frac{y(4)}{a'}} + O\left(\frac{\log n}{n}\right)\right]. \quad \textbf{[14]}$$

COROLLARY 1. *The result of* Theorem 2 *holds with an additional error term* $O(1/n)$ *for a nongaussian harmonizable process with finite fourth-order moments if the fourth-order cumulants satisfy*

$$\frac{1}{n}\sup_t \sum_{\tau,t',\tau'=-n/2}^{n/2} |\operatorname{cum}(x_t, x_\tau, x_{t'}, x_{\tau'})| \to 0. \quad \textbf{[15]}$$

This will be the case if

$$\sup_t \sum_{\tau,t',\tau'=-\infty}^{\infty} |\operatorname{cum}(x_t, x_\tau, x_{t'}, x_{\tau'})| < \infty. \quad \textbf{[16]}$$

Spectral Estimates

We consider an estimate $\hat f_{\alpha,\omega}(\eta)$ of $\tilde f_{\alpha,\omega}(\eta)$ obtained by smoothing the periodogram. Let $K(\eta)$ be a nonnegative bounded weight function of finite support with $\int K(\eta)dy = 1$. The weight function $K_n(\eta) = b_n^{-1} K(b_n^{-1}\eta)$ with $b_n \downarrow 0$ as $n \to \infty$ and $nb_n \to \infty$. The weight functions K_n should be considered as functions on the circle $(-\pi, \pi]$ with $-\pi$ identified with π. The estimate

$$\hat f_{\alpha,\omega}(\eta) = \int_{-\pi}^{\pi} I_n(\alpha\mu+\omega,\mu)K_n(\mu-\eta)d\mu. \quad \textbf{[17]}$$

PROPOSITION 2. *If* A2 *is strengthened so that the spectral densities* $f_{a_i,b_i}(\nu)$ *are assumed to be twice continuously differentiable and* K *is symmetric, then*

$$E\hat f_{\alpha,\omega}(\eta) - \tilde f_{\alpha,\omega}(\eta) = O(b_n^2) + O\left(\frac{\log n}{n}\right)$$

where

$$\tilde f_{\alpha,\omega}(\eta) = \sum_{y(k,a,b)=0} f_{a,b}(\eta+2\pi k)\frac{\min(a,1)}{a}$$

as $n \to \infty$.

The asymptotic behavior of covariances is described in the following result.

THEOREM 3. *Let* X_t *be a continuous time parameter harmonizable process continuous in mean square satisfying assumptions* A1–A3 *and* **16**. *Then*

$$\operatorname{cov}(\hat f_{\alpha,\omega}(\eta), \hat f_{\alpha',\omega'}(\eta')) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} K_n(\mu-\eta)K_n(\mu'-\eta') \times \operatorname{cov}(I_n(\alpha\mu+\omega,\mu), I_n(\alpha'\mu'+\omega',\mu'))d\mu d\mu'$$

$$= \frac{2\pi}{nb_n}\left[\sum{}' \frac{\min(a',1)}{a\alpha'}\min\left(\alpha'\frac{\min(a,1)}{\min(a',1)},1\right) \times f_{a,b}(\alpha'\eta'+\omega'+2\pi k)f_{a',b'}(-\eta'+2\pi k') \times \int K(a'y)K(y)dy + \sum{}'' \frac{\alpha'\min(a',1)}{a} \times \min\left(\frac{1}{\alpha'}\frac{\min(a,1)}{\min(a',1)},1\right) \times f_{a,b}(-\eta'+2\pi k)f_{a',b'}(\alpha'\eta'+\omega'+2\pi k') \times \int K(-\alpha'a'y)K(y)dy\right] + o\left(\frac{1}{nb_n}\right),$$

where the first sum Σ' *is over* a, a', b, b', k, k' *such that*

$$a\alpha' = a'\alpha,\ (2\pi ka + a\omega' + b - \omega)\bmod 2\pi = -\alpha((2\pi k'a' + b')\bmod 2\pi),\ -2\pi k'a' + 2\pi j' + a'\eta' - \eta - b' = 0$$

with $2\pi j' = 2\pi k'a' + b' - ((2\pi k'a' + b')\bmod 2\pi)$, *while the second sum* Σ'' *is over* a, a', b, b', k, k' *such that*

$$a = \alpha a'\alpha,\ (2\pi ka - \omega + b)\bmod 2\pi = -\alpha((2\pi k'a' + a'\omega' + b')\bmod 2\pi),\ -2\pi k'a' - a'\alpha'\eta' - \eta - \alpha'\omega - b' + 2\pi j' = 0$$

with

$$2\pi j' = 2\pi k'a' + a'\alpha'\eta' + \eta + a'\omega' + b' - ((2\pi k'a' + a'\alpha'\eta + a'\omega' + b')\bmod 2\pi).$$

In the almost periodic case we have the following corollary.

COROLLARY 2. *If the assumptions of* Theorem 3 *are satisfied with X_t almost periodic*

$$\mathrm{cov}(\hat{f}_{\omega}(\eta), \hat{f}_{\omega'}(\eta')) = \frac{2\pi}{nb_n}\left[\sum{}' f_b(\eta' + \omega' + 2\pi k) f_{b'}(-\eta' + 2\pi k') \times \int K^2(y)dy + \sum{}'' f_b(-\eta' + 2\pi k) f_{b'}(\eta' + \omega' + 2\pi k') \int K(-y)K(y)dy\right] + o\left(\frac{1}{nb_n}\right)$$

where the first sum is over b, b', k, k' such that

$$(\omega' - \omega + b)\bmod 2\pi = (-(b')\bmod 2\pi) - 2\pi k' + 2\pi j' + \eta' - \eta - b' = 0$$

with $2\pi j' = 2\pi k' + b' - ((b')\bmod 2\pi)$ *while the second sum is over b, b', k, k' with*

$$(-\omega + b)\bmod 2\pi = -(\omega' + b')\bmod 2\pi, \ -2\pi k' - \eta' - \eta - \omega - b' + 2\pi j' = 0$$

with

$$2\pi j' = 2\pi k' + \eta' + \eta + \omega' + b' - ((\eta' + \eta + \omega' + b)\bmod 2\pi).$$

On heuristic grounds one would expect to be able to estimate a spectral density localized on a piecewise smooth curve in the plane.

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