Short Range and Long Range Dependence

Murray Rosenblatt

1 Introduction

In this section a discussion of the evolution of a notion of strong mixing as a measure of short range dependence and with additional restrictions a sufficient condition for a central limit theorem, is given. In the next section I will give a characterization of strong mixing for stationary Gaussian sequences. In Sect. 3 I will give a discussion of processes subordinated to Gaussian processes and in Sect. 4 results concerning the finite Fourier transform is noted. In Sect. 5 a number of open questions are considered.

In an effort to obtain a central limit theorem for a dependent sequence of random variables in [12], I made use of a blocking argument of S.N. Bernstein [1] and was led to what I called a strong mixing condition [2, 12]. In the blocking argument big blocks are separated by small blocks. Consider a sequence of random variables X_n, $n = \ldots, -1, 0, 1, \ldots$. Let $\mathscr{B}_n$ and $\mathscr{F}_m$ be the σ-fields generated by $X_j, j \leq n$ and X_j, $j \geq m$, respectively. If

$$\sup_{A \in \mathscr{B}_n, B \in \mathscr{F}_m} |P(A \cap B) - P(A)P(B)| \leq \alpha(m-n),$$

$m > n$ with $\alpha(k) \to 0$ as $k \to \infty$, the sequence $\{X_n\}$ is said to satisfy a *strong mixing condition*. Such a sequence needn't be stationary. A sequence with such a strong mixing condition can be thought of as one with short range dependence and its absence an indicator of long range dependence.

M. Rosenblatt (✉)
Department of Mathematics, University of California, San Diego,
La Jolla, CA 92093-0112, USA
e-mail: mrosenblatt@math.ucsd.edu

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The strong mixing condition together with the following assumptions are enough to obtain asymptotic normality for partial sums of the sequence. Assume that $EX_n = 0$ for all n. The critical additional assumptions are

1.
$$E\left|\sum_{j=a}^{b} X_j\right|^2 \sim h(b-a)$$
as $b-a \to \infty$ with $h(m) \uparrow \infty$ as $m \to \infty$, where $x(\theta) \sim y(\theta)$ means $x(\theta)/y(\theta) \to 1$ as $\theta \to \theta_0$ and

2.
$$E\left|\sum_{j=a}^{b} X_j\right|^{2+\delta} = O\left(h(b-a)\right)^{1+\delta/2}$$
as $b-a \to \infty$ for some $\delta > 0$.

The following theorem was obtained.

Theorem 1. *If $\{X_n\}$, $E(X_n) = 0$, is a sequence satisfying a strong mixing condition and assumptions 1. and 2., we can determine numbers k_n, p_n, q_n satisfying*

$$k_n(p_n + q_n) = n,$$
$$k_n, p_n, q_n \to \infty,$$
$$q_n/p_n \to 0$$

as $n \to \infty$ such that

$$\frac{S_n}{\sqrt{k_n \cdot h(p_n)}} \qquad (S_n = \sum_{j=1}^{n} X_j)$$

is asymptotically normally distributed with positive variance (see [12] and [2]).

In the argument the numbers $k_n \alpha(q_n)$ have to be made very small. An elegant statement of a result can be given in a stationary case (see Bradley [3] for a proof).

Theorem 2. *Let $\{X_n\}$ be a strictly stationary sequence with $E(X_0) = 0$, $EX_0^2 < \infty$ that is strongly mixing and let $\sigma_n^2 = ES_n^2 \to \infty$ as $n \to \infty$. The sequence (S_n^2/σ_n^2) is uniformly integrable if and only if S_n/σ_n is asymptotically normally distributed with mean zero and variance one.*

In the paper [9] Kolmogorov and Rozanov showed that a sufficient condition for a stationary Gaussian sequence to be strongly mixing is that the spectral distribution be absolutely continuous with positive continuous spectral density.

One should note that the concept of strong mixing here is more restrictive in the stationary case than the ergodic theory concept of strong mixing. A very extensive discussion of our notion of strong mixing as well as that of other related concepts is given in the excellent three volume work of Richard Bradley [3].

In 1961 corresponding questions were taken up for what is sometimes referred as narrow band-pass filtering in the engineering literature. These results are strong enough to imply asymptotic normality for the real and imaginary parts of the truncated Fourier transform of a continuous time parameter stationary process. Let $X(t)$, $EX(t) = 0$, be a separable strongly mixing stationary process with $EX^4(t) < \infty$ that is continuous in mean of fourth order. If the covariance and 4th order cumulant function are integrable, it than follows that

$$\left(\frac{1}{2}T\right)^{-1/2} \int_0^T \cos(\lambda t) X(t) dt,$$

$$\left(\frac{1}{2}T\right)^{-1/2} \int_0^T \sin(\lambda t) X(t) dt, \qquad \lambda \neq 0,$$

are asymptotically normal with variance

$$\pi f(\lambda)$$

and independent as $T \to \infty$ ($f(\lambda)$ the spectral density of $X(t)$ at λ). This follows directly from the results given in [14].

2 Gaussian Processes

In the 1961 paper [13] a Gaussian stationary sequence $\{Y_k\}$ with mean zero and covariance

$$r_k = EY_0 Y_k = (1 + k^2)^{-D/2} \sim k^{-D} \qquad \text{as} \quad k \to \infty,$$

$0 < D < 1/2$, was considered. The normalized partial sums process

$$Z_n = n^{-1+D} \sum_{k=1}^{n} X_k$$

of the derived quadratic sequence

$$X_k = Y_k^2 - 1$$

was shown to have a limiting non-Gaussian distribution as $n \to \infty$. The characteristic function of the limiting distribution is

$$\phi(\theta) = \exp\left(\frac{1}{2}\sum_{k=2}^{\infty}(2i\theta)^k c_k/k\right)$$

with

$$c_k = \int_0^1 dx_1 \cdots \int_0^1 dx_k |x_1 - x_2|^{-D}|x_2 - x_3|^{-D} \cdots |x_{k-1} - x_k|^{-D}|x_k - x_1|^{-D}.$$

Since conditions 1. and 2. are satisfied by X_k, the fact that the limiting distribution is non-Gaussian implies that $\{X_k\}$ and $\{Y_k\}$ cannot be strongly mixing.

In their paper Helson and Sarason [6] obtained a necessary and sufficient condition for a Gaussian stationary sequence to be strongly mixing. This was that the spectral distribution of the sequence be absolutely continuous with spectral density w

$$w = |P|^2 \exp(u + \tilde{v})$$

with P a trigonometric polynomial and u and v real continuous functions on the unit circle and $\tilde{v}$ the conjugate function of v.

It is of some interest to note that the functions of the form

$$\exp(u + \tilde{v}) = w,$$

with u and w continuous are such that w^n is integrable for every positive or negative integer n. (The set of such functions w is W.) An example with a discontinuity at zero is noted in Ibragimov and Rozanov [7]

$$f(\lambda) = \exp\left\{\sum_{k=1}^{\infty}\frac{\cos(k\lambda)}{k(\ln k + 1)}\right\}.$$

Making use of results on trigonometric series with monotone coefficients, it is clear that

$$\sum_{k=1}^{\infty}\frac{\sin(k\lambda)}{k(\ln k + 1)}$$

is continuous and that

$$\ln f(\lambda) \sim \ln\ln\frac{1}{\lambda}$$

as $\lambda \to 0$. So, $f(\lambda)$ and $1/f(\lambda)$ are both spectral densities of strongly mixing Gaussian stationary sequences. $f(\lambda)$ has a discontinuity at $\lambda = 0$ while $1/f(\lambda)$ is continuous with a zero at $\lambda = 0$. Sarason has also shown in [15, 16] that the functions $\log w$, $w \in W$, have vanishing mean oscillation. Let f be a complex function on $(-\pi, \pi]$ and I an interval with measure $|I|$.

Let

$$f_I = |I|^{-1} \int_I f(x)dx$$

and

$$M_a(f) = \sup_{|I| \le a} |I|^{-1} \int_I |f(x) - f_I| dx.$$

f is said to be of bounded mean oscillation if $M_{2\pi}(f) < \infty$. Let

$$M_0(f) = \lim_{a \to 0} M_a(f).$$

f is said to be of vanishing mean oscillation if f is of bounded mean oscillation and $M_0(f) = 0$.

In the case of a vector valued (d-vector) stationary strong mixing Gaussian sequence there is $d_0 \le d$ such that the spectral density matrix $w(\lambda)$ has rank d_0 for almost all λ. If $d_0 = d$ the sequence is said to have full rank. The case of sequences of rank $d_0 < d$ can be reduced to that of sequences of full rank. A result of Treil and Volberg [20] in the full rank case is noted.

Theorem 3. *Assume that the spectral density w of a stationary Gaussian process is such that $w^{-1} \in L^1$. The process is strongly mixing if and only if*

$$\lim \sup_{|\lambda| \to 1} \left\{ \det(w(\lambda)) \exp\left(-[\log \det w](\lambda)\right) \right\} = 1,$$

where $\det(w(\lambda))$ *and* $[\log \det w](\lambda)$ *are the harmonic extensions of w and* $\log \det w$ *on the unit circle at the point λ on the unit disc.*

The harmonic extension u on the unit disc of a function f on the unit circle is given via the Poisson kernel

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2} = \mathrm{Re}\left(\frac{1 + re^{i\theta}}{1 - re^{i\theta}}\right), \qquad 0 \le r \le 1,$$

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) f(e^{it}) dt.$$

3 Processes Subordinated to Gaussian Processes

In the paper [18] M. Taqqu considered the weak limit of the stochastic process

$$Z_n(t) = n^{-1+D} \sum_{k=1}^{[nt]} X_k$$

as $n \to \infty$ and noted various properties of the limit process. Here $[s]$ denotes the greatest integer less than or equal to s. M. Taqqu [19] and R. Dobrushin and P. Major [5] discovered about the same time that the simple example of M. Rosenblatt was a special case of an interesting broad class of nonlinear processes subordinated to the Gaussian stationary processes. Consider $\{X_n\}$, $EX_n = 0$, $EX_n^2 = 1$ a stationary Gaussian sequence with covariance

$$r(n) = n^{-\alpha} L(n), \qquad 0 < \alpha < 1,$$

where $L(t)$, $t \in (0, \infty)$ is slowly varying. Let $H(\cdot)$ be a function with

$$EH(X_n) = 0, \qquad EH^2(X_n) = 1.$$

$H_j(\cdot)$ is the jth Hermite polynomial with leading coefficient one. Then $H(\cdot)$ can be expanded in terms of the H_j's

$$H(X_n) = \sum_{j=1}^{\infty} c_j H_j(X_n)$$

with

$$\sum_{j=1}^{\infty} c_j^2 j! < \infty.$$

Assume that $\alpha < 1/k$ with k the smallest index such that $c_k \neq 0$ (H is then said to have rank k). Set

$$A_N = N^{1-k\alpha/2} (L(N))^{k/2}$$

and

$$Y_n^N = A_N^{-1} \sum_{j=N(n-1)}^{Nn-1} H(X_j),$$

$n = \ldots, -1, 0, 1, \ldots$ and $N = 1, 2, \ldots$. Then the finite dimensional distributions of Y_n^N, $n = \ldots, -1, 0, 1, \ldots$ as $N \to \infty$ tend to those of the sequence Y_n^*

$$Y_n^* = d^{-k/2} c_k \int e^{in(x_1+\cdots+x_k)} \frac{e^{i(x_1+\cdots+x_k)} - 1}{i(x_1 + \cdots + x_k)} |x_1|^{\frac{\alpha-1}{2}} \cdots |x_k|^{\frac{\alpha-1}{2}} dW(x_1) \cdots dW(x_k)$$

with $W(\cdot)$ the Wiener process on $(-\infty, \infty)$ where in the integration it is understood that the hyper-diagonals $x_i = x_j, i \neq j$ are excluded, and

$$d = \int \exp(ix)|x|^{\alpha-1} dx = 2\Gamma(x) \cos\left(\frac{\alpha\pi}{2}\right).$$

In [4] P. Breuer and P. Major obtained central limit theorems for nonlinear functions of Gaussian stationary fields. As in the discussion of results for noncentral limit theorem we shall consider the case of stationary sequences. Again, let

$$Y_n^N = A_N^{-1} \sum_{j=N(n-1)}^{Nn-1} H(X_j),$$

with X_n a stationary Gaussian sequence $EX_n = 0$, $EX_n^2 = 1$. $H(\cdot)$ is real-valued with

$$EH(X_n) = 0, \qquad EH^2(X_n) < \infty.$$

Assume that H has rank k and that

$$\sum_n |r(n)|^k < \infty$$

($r(\cdot)$ the covariance function of the X sequence). Let H_l be the lth Hermite polynomial. With $A_N = N^{1/2}$ the limits

$$\lim_{N\to\infty} E\left(Y_0^N(H_l)\right)^2 = \lim_{N\to\infty} A_N^{-2} l! \sum_{-N \le i,j < 0} r^l(i-j) = \sigma_l^2 l!$$

exist for all $l \geq k$ and

$$\sigma^2 = \sum_{l=k}^{\infty} c_l^2 l! \sigma_l^2 < \infty.$$

The finite dimensional distributions of Y_n^N as $N \to \infty$ tend to the finite dimensional distributions of σZ_n with the Z_n i.i.d. standard normal random variables. T.C. Sun obtained the case of this result for $k = 2$ in [17].

4 Finite Fourier Transform

In 1961 paper [14] I showed that in the case of a separable continuous time parameter process a variety of filters amounting to narrow band-pass filtering, under the assumption of strong mixing, integrability of the covariance function and the 4th order cumulant function, stationarity and positivity of the spectral density imply asymptotic normality. This implies that

$$\int_0^T \cos(\lambda t) X(t) dt, \qquad \int_0^T \sin(\lambda t) X(t) dt$$

are asymptotically normal as $T \to \infty$ for all λ and independent for $\lambda \neq 0$ as $T \to \infty$.

A recent paper of Peligrad and Wu [11] is of considerable interest. They use a stationary ergodic Markov sequence ξ_n on the probability space $(\Omega, \mathscr{F}, P)$ with marginal distribution

$$\pi(A) = P(\xi_0 \in A).$$

Let

$$\mathscr{L}_0^2(\pi) = \left\{ h : \int h^2 d\pi < \infty, \int h d\pi = 0 \right\},$$

$$\mathscr{F}_k = \mathscr{B}\{\xi_j, j \leq k\}, \qquad X_j = h(\xi_j).$$

The condition

$$E(X_0 | \mathscr{F}_{-k}) = 0 \qquad P \text{ almost surely} \tag{1}$$

is of particular interest. They obtain the following theorem among others.

Theorem 4. *If (X_k) is stationary ergodic satisfying (1) then for almost all $\theta \in (0, 2\pi)$*

$$\lim_{n \to \infty} \frac{E|S_n(\theta)|^2}{n} = g(\theta), \qquad S_n(\theta) = \sum_{k=1}^{[n\theta]} X_k$$

with g integrable over $[0, 2\pi]$ and

$$\frac{1}{\sqrt{n}} [\mathrm{Re}(S_n(\theta)), \mathrm{Im}(S_n(\theta))] \Rightarrow [N_1(\theta), N_2(\theta)]$$

under P with $S_n(\theta)$ the Fast Fourier transform computed at θ and $N_1(\theta)$, $N_2(\theta)$ independent identically distributed normal random variables with mean zero and variance $g(\theta)/2$. One can always take X_k as a function of a Markov sequence $\xi_n = (X_k, k \leq n)$.

In a number of examples one considers derived sequences

$$Z_n^N = A_N^{-1} \sum_{j \in B_n^N} \xi_j \qquad N = 1, 2, \ldots,$$

with

$$B_n^N = \{j : nN \le j < (n+1)N\}$$

and A_N a norming constant (which needn't be $\sqrt{N}$). The interest is in convergence of the finite dimensional distributions of the sequence Z_n^N as $N \to \infty$ to finite dimensional distributions of a limit sequence Z_n^*. The object is to determine the appropriate norming constant A_N and the character of the nontrivial limit sequence Z_n^*. One is also led to the following question – for which sequences ξ_n does one have

$$(\xi_{n_1}, \ldots, \xi_{n_k}) \stackrel{\mathrm{d}}{=} (Z_{n_1}^N, \ldots, Z_{n_k}^N)$$

(equality in distribution for all $N = 1, 2, \ldots$ and $n_1, \ldots, n_k$). If this is satisfied with $A_N = N^\alpha$, ξ_n is said to be a *self-similar* sequence with self-similarity parameter α.

In the case of the limit theorems of Taqqu [18, 19], Dobrushin and Major [5] the limit processes are self-similar with self-similarity parameter α.

It's of interest to note that if the covariances

$$r(n) = n^{-\alpha} L(n), \qquad \alpha \in (0, 1)$$

with $L(n)$ slowly varying are monotone

$$f_\alpha(x) = \sum_{n=1}^{\infty} r(n) \cos nx,$$

$$g_\alpha(x) = \sum_{n=1}^{\infty} r(n) \sin nx,$$

converge uniformly outside an arbitrarily small neighbourhood of $x = 0$ and

$$f_\alpha(x) \sim x^{\alpha-1} L(x^{-1}) \Gamma(1-\alpha) \sin\left(\frac{1}{2}\pi\alpha\right),$$

$$g_\alpha(x) \sim x^{\alpha-1} L(x^{-1}) \Gamma(1-\alpha) \cos\left(\frac{1}{2}\pi\alpha\right)$$

as $x \to 0+$. The real spectral density of the Gaussian stationary process with covariances $r(n)$ has a singularity at $x = 0$. Given the Hermite polynomial H_k consider the derived process $H_k(X_j)$ (X_j the Gaussian process). The covariance of the derived process

$$EH_k(X_0)H_k(X_j) = k!r(j)^k$$

so its spectral density will have a singularity at zero if and only if $k\alpha < 1$.

A limit theorem of Kesten and Spitzer [8] is of great interest.

$$S_n = X_1 + \cdots + X_n, \qquad n \geq 1,$$

is the simple random walk on the integers ($X_i = \pm 1$ with probability 1/2 and i.i.d.) with random sequence $\xi(x)$, x integer, i.i.d. with the same distribution as the X_i's but independent of them. The asymptotic behaviour of

$$U_n = \sum_{k=1}^{n} \xi(S_k)$$

is considered as $n \to \infty$. U_s is the linearly interpolated process. They show that

$$n^{-3/4} U_{nt}, \qquad t \geq 0,\ n = 1, 2, 3, \ldots$$

converges weakly to

$$\int_{-\infty}^{\infty} L_t(x) dZ(x), \qquad t \geq 0,$$

where $L_t(x)$ is the local time at x of Brownian motion B_t and $Z(x)$ is a Brownian motion with time $-\infty < x < \infty$.

$\xi(S_k)$, $k = 1, 2, \ldots$ can be extended to a two-sided stationary sequence as follows. Introduce $X_0, X_{-1}, X_{-2}, \ldots$ as i.i.d. random variables with the same distribution as the earlier random variables and independent of all the other variables. Let $\eta_0 = \xi(0)$,

$$\eta_i = \begin{cases} \xi\left(\sum_{j=0}^{i-1} X_j\right) & \text{if } i > 0 \\ \xi\left(-\sum_{j=-1}^{i} X_j\right) & \text{if } i < 0 \end{cases}.$$

The sequence η_i is stationary and we obtain an approximation to its spectral density

$$E(\eta_0 \eta_i) = \begin{cases} 0 & \text{if } \sum_{j=0}^{i-1} X_j \neq 0\,, \quad i > 0 \\ E(\xi^2(0))\binom{2m}{m}\frac{1}{2^{2m}} & \text{if } \sum_{j=0}^{i-1} X_j = 0\,, \quad i = 2m \end{cases}$$

and

$$2^{-2m}\binom{2m}{m} \sim \frac{2^{1/2}}{\sqrt{2\pi}} \frac{1}{\sqrt{m}}$$

as $m \to \infty$. This suggests that the spectral density is of the form

$$\sum_m \frac{1}{\sqrt{m}} \cos mx$$

and this behaves like

$$(2\lambda)^{-1/2}\Gamma(1/2) \sim \frac{\pi}{4}$$

as $\lambda \to 0$.

5 Open Questions

The almost everywhere character (in θ) of the result of Peligrad and Wu indicates that the asymptotics of the finite Fourier transform at points where there is a singularity of the spectral density functions are not dealt with. This would, for example, be the case if we had a Gaussian stationary sequence (X_j) with covariance of the form

$$r(n) = \sum_j \beta_j |n|^{-\alpha_j} \cos n(\lambda - \lambda_j) L_j(n),$$

$\beta_j > 0$, $0 < \alpha_j < 1$, λ_j distinct, and wished to compute the finite Fourier transform of $H(X_k)$ at $\lambda = \lambda_j$ with the leading non-zero Fourier-Hermite coefficient k of $H(\cdot)$ such that $k\alpha_j < 1$. As before the $L_j(\cdot)$ are slowly varying. The variance of the finite Fourier transform and its limiting distributions when properly normalized as N tends to infinity are not determined. Of course this is just a particular example of interest under the assumptions made in the theorem of Peligrad and Wu.

The random sequences with covariances almost periodic functions contain a large class of interesting nonstationary processes. The harmonizable processes of this type have all their spectral mass concentrated on at most a countable number of

$$\lambda = \mu + b, \quad b = b_j\ , j = \ldots, -1, 0, 1, \ldots.$$

It would be of some interest to see whether one could characterize the Gaussian processes of this type which are strongly mixing. Assume that the spectra on the lines of support are absolutely continuous with spectral densities $f_b(u)$. Under rather strong conditions one can estimate the $f_b(\cdot)$ (see [10]). However, there are still many open questions.

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