COSECTION LOCALIZATION AND VANISHING FOR VIRTUAL FUNDAMENTAL CLASSES OF D-MANIFOLDS

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Abstract. We establish cosection localization and vanishing results for virtual fundamental classes of derived manifolds, combining the theory of derived differential geometry by Joyce with the theory of cosection localization by Kiem-Li. As an application, we show that the stable pair invariants of hyperkähler fourfolds, defined by Cao-Maulik-Toda, are zero.

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1. Introduction

The study of coherent sheaves and vector bundles on Calabi-Yau manifolds has been a long-standing subject of interest in mathematics and theoretical physics.

From the point of view of enumerative geometry, the virtual counts of coherent sheaves on Calabi-Yau threefolds are known as Donaldson-Thomas invariants and were introduced by Thomas [Tho00]. He showed that moduli spaces of stable sheaves admit a perfect obstruction theory [BF97] and thus a virtual fundamental cycle [BF97, LT98]. These moduli spaces enjoy many rich structures and the invariants and their generalizations and refinements have been extensively studied (for a far from exhaustive list of references, cf. [Beh09, JS12, KS10, BBJ19, BBD+15, Oko19, KLS17, Sav20, KS20]).

Suppose now that $W$ is a smooth, projective complex Calabi-Yau fourfold (meaning that its canonical bundle is trivial, $K_W \cong \mathcal{O}_W$) and $X$ a proper moduli scheme parameterizing stable sheaves on $W$. In this case, the deformations and (higher) obstructions for a sheaf $E$ on $W$ are controlled by the groups

$$\text{Ext}^1(E, E), \text{Ext}^2(E, E), \text{Ext}^3(E, E)$$
and thus the natural candidate complex for a perfect obstruction theory has three terms, making the results of [Beh09, LT98] and the theory of virtual fundamental cycles not directly applicable.

In order to define Donaldson-Thomas invariants, Cao-Leung [CC14] first suggested a gauge-theoretic approach, which produces invariants in certain cases. Later, in their seminal paper [BJ17], Borisov-Joyce generalized this approach to define invariants in great generality, making crucial use of the fact that $X$ is the truncation of a $(-2)$-shifted symplectic derived scheme [PTVV13] and using the theory of derived differential geometry [Joy]. The virtual fundamental classes they define are thus of differential and not algebraic nature and depend on a choice of orientation, which exists by [CGJ20]. In forthcoming work [OTa, OTb], Oh-Thomas develop a definition strictly within the realm of algebraic geometry and prove that their virtual fundamental cycle coincides with the class of Borisov-Joyce.

Computing Donaldson-Thomas invariants and invariants of Donaldson-Thomas type for Calabi-Yau fourfolds has attracted significant interest recently, cf. [CK18, CK19, CKM19, CKM20, CT19, CT20, Cao18, CMT19, CMT18].

One of the most important methods in handling such invariants in algebraic geometry has been the localization of virtual fundamental cycles by cosections introduced by Kiem-Li [KL13].

The purpose of this paper is to establish cosection localization and vanishing results for the virtual fundamental class appearing in the context of Borisov-Joyce, which will aid in computing these invariants. Since this is not algebraic, to do so, we combine the theory of derived manifolds by Joyce [Joy] with the topological version of cosection localization by Kiem-Li [KL18].

We hope to develop a corresponding (more uniform and stronger) formalism of cosection localization for the algebraic virtual cycle of Oh-Thomas in the future and this will be a subject of further study.

Statement of results. Here we give brief statements of our main results, which fall into two categories: vanishing of the virtual fundamental class in the presence of a surjective cosection and localization by complex cosections which “come from geometry”.

We have the following vanishing results.

**Theorem** (Theorem 4.4). Let $\mathcal{X}$ be a compact, oriented $d$-manifold with a continuous family of surjective $\mathbb{R}$-linear maps $\sigma_x: h^1(T\mathcal{X}|_x) \to \mathbb{R}$ and underlying topological space $X$. Then its virtual fundamental class is zero, i.e. $[\mathcal{X}]^{\text{vir}} = 0 \in H_{\text{vir}, \text{dim}}(X)$.

This statement is essentially already embedded in the literature, following directly from the definition of the virtual fundamental class of $\mathcal{X}$ given in [Joy].

In [BJ17], the virtual fundamental class of a $(-2)$-shifted derived scheme $(\mathcal{X}, \omega_\mathcal{X})$ is obtained by truncating $\mathcal{X}$ in a suitable way to produce an associated $d$-manifold $\mathcal{X}_{\text{dim}}$ and then taking its virtual fundamental class. In this setting, using the above theorem gives the following vanishing for surjective cosections that are non-degenerate with respect to the symplectic form.
Theorem (Theorem 4.8). Let \((X, \omega_X)\) be a proper, oriented \((-2)\)-shifted derived scheme with a non-degenerate cosection \(\sigma: T|_X \to O_X\) (cf. Definition 4.6). Then its virtual fundamental class vanishes, i.e. \([\mathcal{X}_{dm}]_{\text{vir}} = 0 \in H_{\text{vir}, \dim}(X)\).

As a corollary, we obtain the vanishing of stable pair invariants of hyperkähler fourfolds.

Corollary (Theorem 6.2, [CMT19, Claim 2.19]). Let \(P_n(W, \beta)\) be the moduli space parameterizing stable pairs \(I = [O_W \to F]\) on a hyperkähler fourfold \(W\) satisfying \([F] = \beta \in H_2(W), \chi(F) = n\). If \(\beta \neq 0\) or \(n \neq 0\), then the virtual fundamental class is zero, i.e. \([P_n(W, \beta)]_{\text{vir}} = 0\).

Finally, when a derived manifold \(X\) and a cosection \(\sigma\) satisfy conditions bringing them closer to being complex analytic or obtained from analytic data, we can localize the virtual fundamental class by the cosection.

**Theorem** (Theorem 5.7). Let \(X\) be a compact, oriented \(d\)-manifold with a complex cosection \(\sigma: h^1(T_X|_X) \to C_X\) such that \(X\) and \(\sigma\) satisfy the conditions of Setup 5.2. Let \(X(\sigma)\) be the locus where \(\sigma\) is not surjective and write \(i: X(\sigma) \to X\) for the inclusion. Then the virtual fundamental class localizes to \(X(\sigma)\), i.e. there exists a cosection localized class \([\mathcal{X}]_{\text{loc}, \sigma}\) satisfying
\[
i_*[\mathcal{X}]_{\text{loc}, \sigma} = [\mathcal{X}]_{\text{vir}} \in H_{\text{vir}, \dim}(X(\sigma)).
\]

### Layout of the paper

In §2, we provide necessary background on \(d\)-manifolds, \((-2)\)-shifted symplectic derived schemes and their relationship. §3 contains material on the topological definition of normal cones and the virtual fundamental class of a \(d\)-manifold. In §4 we establish vanishing results for surjective real cosections and in §5 we treat the case of complex cosections coming from geometry. Finally, in §6 we obtain the vanishing of stable pair invariants of hyperkähler fourfolds.

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### Notation and conventions

We work with Borel-Moore homology and cohomology with integer coefficients. \(\mathcal{C}^\infty\)-schemes are separated, second countable and locally fair. All algebraic and derived schemes are defined over the field of complex numbers \(\mathbb{C}\). Classical and derived algebraic schemes and complex analytic spaces are separated and of finite type.

\(\mathcal{X}\) will typically denote a derived scheme and \(\omega_{\mathcal{X}}\) a \((-2)\)-shifted symplectic form on \(\mathcal{X}\). \(\mathcal{X}\) will typically denote a \(d\)-manifold and \(X_{\mathcal{C}^\infty}\) its underlying \(\mathcal{C}^\infty\)-scheme. \(X\) will always denote the underlying topological space of \(\mathcal{X}\) or \(\mathcal{X}\) and sometimes, by abuse of notation, the classical truncation of \(\mathcal{X}\) as well. \(X^{an}\) denotes the complex analytic space associated to a scheme \(X\).

2. **Background on \(d\)-Manifolds and \((-2)\)-Shifted Symplectic Derived Schemes**

In this section, we collect some requisite background and introduce terminology that will be used in the rest of the paper.
2.1. $\mathcal{C}^\infty$-schemes and d-manifolds. The main references for the theory of $\mathcal{C}^\infty$-schemes and d-manifolds are the books [Joy19] and [Joy] by Joyce. Here we give a brief informal account of their structure.

Roughly speaking, under the conventions of this paper, a $\mathcal{C}^\infty$-scheme is a locally ringed space that is locally isomorphic to the (real) spectrum of the $\mathcal{C}^\infty$-ring $\mathcal{C}^\infty(\mathbb{R}^n)/I$, where $I$ is an ideal of smooth functions on $\mathbb{R}^n$. There is a theory of (quasi)coherent sheaves and vector bundles on $\mathcal{C}^\infty$-schemes, which resembles that of sheaves on algebraic schemes.

$\text{d}$-manifolds form a theory of derived differential geometry being analogous to quasi-smooth derived or dg-schemes. Their local structure is given as follows.

**Definition 2.1.** [Joy, Example 1.4.4] A principal d-manifold $\mathcal{X} = \mathcal{S}_{Y,E,s}$ is determined by the data of a smooth manifold $Y$, a smooth vector bundle $E \to Y$ and $s \in \mathcal{C}^\infty(E)$ as follows: Its underlying $\mathcal{C}^\infty$-scheme $X_{\mathcal{C}^\infty}$ is the affine scheme obtained as the spectrum of the $\mathcal{C}^\infty$-ring $\mathcal{C}^\infty(Y)/I$, where $I = (s)$ is the ideal generated by the section $s$. The higher data are given by the morphism $E^\vee|X_{\mathcal{C}^\infty} \xrightarrow{s^\vee} I/I^2$ of $\mathcal{O}_{X_{\mathcal{C}^\infty}}$-modules, so that the underlying topological space $\mathcal{X}$ is the zero locus of the section $s$. The tangent complex (or virtual tangent bundle) of $\mathcal{X}$ is the two-term complex

$$T_{\mathcal{X}} = [T_Y|X_{\mathcal{C}^\infty} \xrightarrow{ds} E|X_{\mathcal{C}^\infty}]$$

where for any choice of connection $\nabla$ on $E^\vee$, $ds$ is the restriction to $X_{\mathcal{C}^\infty}$ of $\nabla s: T_Y \to E$.

In general, every d-manifold $\mathcal{X}$ is locally equivalent to a principal d-manifold. When $\mathcal{X}$ is compact (i.e. its underlying topological space is compact), Joyce proves a Whitney embedding theorem, which in particular implies that this is true globally. Since we will be fundamentally interested in compact d-manifolds in this paper, this allows us in practice to consider the case of principal d-manifolds, which are a considerable simplification.

**Theorem-Definition 2.2.** [Joy, Theorems 4.29, 4.34] Let $\mathcal{X}$ be a compact d-manifold. Then $\mathcal{X} \simeq \mathcal{S}_{Y,E,s}$ where $Y$ is an open subset of some Euclidean space $\mathbb{R}^N$. We refer to this equivalence as a presentation of $\mathcal{X}$ as a principal d-manifold. The virtual dimension of $\mathcal{X}$ is then equal to $N - \text{rk} E$.

In analogy with the case of quasi-smooth derived schemes or schemes with a perfect obstruction theory [BF97], we give the following definition.

**Definition 2.3.** The obstruction sheaf of $\mathcal{X}$ is the coherent sheaf $\mathcal{O}_{\text{b}}\mathcal{X} := h^1(T_{\mathcal{X}}) \in \text{Coh}(X_{\mathcal{C}^\infty})$, where $T_{\mathcal{X}}$ is the tangent complex of $\mathcal{X}$.

Finally we mention that there is a notion of orientation for a d-manifold $\mathcal{X}$, which in the case of a principal d-manifold $\mathcal{S}_{Y,E,s}$ roughly amounts to an orientation of the determinant line bundle $\text{det} T_{\mathcal{X}}$. As in the case of manifolds, compact, oriented d-manifolds admit a virtual fundamental class.

**Theorem 2.4.** [Joy, Section 13] Let $\mathcal{X}$ be a compact, oriented d-manifold with underlying topological space $X$. Then there exists a virtual fundamental class $[\mathcal{X}]^{\text{vir}} \in H_{\text{vir}, \text{dim}}(X)$. Moreover, $[\mathcal{X}]^{\text{vir}}$ only depends on the bordism class of $\mathcal{X}$.
2.2. \((-2)\)-shifted symplectic derived schemes. In this subsection, we establish some terminology and describe local structure results that will be used in the rest of the paper. For an introduction to derived algebraic geometry, we refer the reader to \cite{Toen}. Shifted symplectic structures were introduced in the seminal paper \cite{PTVV}.

In a nutshell, a derived scheme \(X\) is locally modelled by the (derived) spectrum \(\text{Spec} A\) where \(A\) is a commutative differential (negatively) graded \(\mathbb{C}\)-algebra. A \((-2)\)-shifted symplectic structure on \(X\) is given by a 2-form \(\omega_X : T_X \wedge T_X \to \mathcal{O}_X[-2]\) which is non-degenerate and is equipped with extra data that make it closed.

A \((-2)\)-shifted symplectic scheme is a pair \((X, \omega_X)\) consisting of a derived scheme \(X\) and a symplectic structure on it. There is also an appropriate notion of orientation on a \((-2)\)-shifted symplectic scheme (cf. \cite{BJ}). Given the above, we introduce the following terminology.

**Definition 2.5.** The obstruction sheaf of \(X\) is the sheaf \(\mathcal{O}_b X = h^1(T_X|_X)\). The obstruction space of \(X\) at a point \(x \in X\) is the vector space \(\mathcal{O}_b(X, x) = h^1(T_X|_x)\).

In particular, the symplectic form \(\omega_X\) induces a family of non-degenerate quadratic forms \(q_x : \mathcal{O}_b(X, x) \to \mathbb{C}\).

We conclude this subsection with a local description of \((-2)\)-shifted symplectic schemes that will be essential later on. In \cite{BBJ19}, the local structure of \((X, \omega_X)\) around every point \(x \in X\) is described by charts in Darboux form as follows.

**Theorem-Definition 2.6.** \cite[Example 5.16]{BBJ19} Let \(x \in X\) be a point of a \((-2)\)-shifted symplectic derived scheme \((X, \omega_X)\). Then there exists a Zariski open neighbourhood \(U = \text{Spec} A \to X\) around \(x\) such that:

1. \((A, \delta)\) is a commutative differential graded algebra, whose degree zero part \(A^0\) is a smooth \(\mathbb{C}\)-algebra of dimension \(m\), with a set of étale coordinates \(\{x_i\}_{i=1}^m\).
2. \(A\) is freely generated over \(A^0\) by variables \(\{y_j\}_{j=1}^n\) and \(\{z_i\}_{i=1}^m\) in degrees \(-1\) and \(-2\) respectively.
3. Let \(\omega_A\) be the pullback of \(\omega_X\) to \(\text{Spec} A\). There exist invertible elements \(q_1, \ldots, q_n \in A^0\) such that
   \[
   \omega_A = \sum_{i=1}^m dz_i dx_i + \sum_{j=1}^n d(q_j y_j) dy_j.
   \]
4. The differential \(\delta\) is determined by the equations
   \[
   \delta x_i = 0, \quad \delta y_j = s_j, \quad \delta z_i = \sum_{j=1}^n y_j \left(2q_j \frac{\partial s_j}{\partial x_i} + s_j \frac{\partial q_j}{\partial x_i}\right)
   \]
   where the elements \(s_j \in A^0\) satisfy
   \[
   q_1 s_1^2 + \ldots + q_n s_n^2 = 0.
   \]

We let \(V = \text{Spec} A^0\), \(E\) the trivial vector bundle of rank \(n\) whose dual has basis given by the variables \(y_j\) and \(F\) the trivial vector bundle whose dual
has basis given by the variables $z_i$. We refer to all of the above data as a derived algebraic Darboux chart for $X$ at $x$.

At the classical level, the truncation $U = h^0(U) \subset X$ is equipped with the following data:

1. A smooth affine scheme $V$ of dimension $m$.
2. A trivial vector bundle $E$ on $V$ of rank $n$ with a non-degenerate quadratic form $q$ which is given by
   \[ q_u(y_1, \ldots, y_n) = q_1(u)y_1^2 + \ldots + q_n(u)y_n^2 \]
   on each fiber of $E$ over $u \in U$.
3. An $q$-isotropic section $s \in \Gamma(E)$ whose scheme-theoretic zero locus is $U \subset V$.
4. A three-term perfect complex of amplitude $[0, 2]$ $G := T U|_U = [T V|_U \xrightarrow{ds} E|_U \xrightarrow{t} F|_U]$.

such that $q_u$ gives a non-degenerate quadratic form on each obstruction space $h^1(G|_u) = \mathcal{Ob}(X, u)$.

We refer to the data $(V, E, G, q, s)$ as an algebraic Darboux chart for $X$ at $x$.

By working in the complex analytic category, since the functions $q_j$ are non-zero at $x$, by possibly shrinking $V$ around $x$, they admit square roots $q_j = r_j^2$ and after re-parametrizing $y_j \mapsto \frac{y_j}{r_j}$ we may assume that $q_j(u) = 1$. We then refer to the data $(V, E, G, q, s)$ as an analytic Darboux chart for $X$ at $x$.

Different Darboux charts glue and thus are compatible through appropriate data of homotopical nature. These compatibilities are important, however we will not need them explicitly. Knowing that they exist and that local constructions play well with them (which is ensured by the results we quote) will be sufficient for our purposes.

2.3. From $(-2)$-shifted symplectic schemes to d-manifolds. Consider a $(-2)$-shifted symplectic derived scheme $(X, \omega_X)$. Let $X = h^0(X)$ be its classical truncation and, by abuse of notation, we write $X$ for the underlying topological space as well.

In this subsection, we briefly recall the main results of [BJ17], where the authors produce a (non-uniquely determined) d-manifold $X^{dm}$ associated to $(X, \omega_X)$ and thus a virtual fundamental class $[X^{dm}]^{vir} \in H_*(X)$.

In order to construct a d-manifold starting with a $(-2)$-shifted symplectic derived scheme, it is necessary to take appropriate (real smooth) truncations of Darboux charts.

**Definition 2.7.** [BJ17, Definition 3.6] Let $(V, E, G, q, s)$ be data of an analytic Darboux chart for $X$ at the point $x$. We say that a real subbundle $E^- \subset E$ is $q$-adapted if for all $u \in U$ the following conditions are met:

1. $\text{im}(ds|_u) \cap E^-|_u = \{0\} \subset E|_u$.
2. $t|_u(E^-|_u) = t|_u(E|_u) \subset F|_u$.
3. The image of the linear map $\Pi_u : \ker(t|_u) \cap E^-|_u \rightarrow \ker(t|_u) \rightarrow h^1(G|_u) = \mathcal{Ob}(X, u)$
is a real maximal negative definite vector subspace of \( \text{Ob}(\mathcal{X}, u) \) with respect to the quadratic form \( \text{Re} q_u \).

Here is a (very) brief outline of the steps involved in the construction of the d-manifold \( X_{dm} \).

**Construction 2.8.**

1. \( (\mathcal{X}, \omega_X) \) admits a cover by derived algebraic Darboux charts, which are compatible in a suitable homotopy-theoretic sense.
2. The classical truncation of these charts gives a cover of \( X \) by analytic Darboux charts.
3. For each such chart \( (V, E, G, q, s) \) at a point \( x \in X \), up to possible shrinking of \( V \) around \( x \), there exists a \( q \)-adapted \( E^- \subset E \). Fix any such choice and let \( E^+ = E/E^- \) and \( s^+ = s \mod E^- \).
4. Using the compatibility data between the derived algebraic Darboux charts, the principal d-manifolds \( S_{V, E^+, s^+} \) glue to a d-manifold \( X_{dm} \) with underlying topological space \( X^{an} \).

Moreover, orientations of \( (\mathcal{X}, \omega_X) \) are in bijection with orientations of \( X_{dm} \). Thus a proper, oriented \( (\mathcal{X}, \omega_X) \) gives rise to a compact, oriented d-manifold \( X_{dm} \) and a virtual fundamental class \( [X_{dm}]^{\text{vir}} \in H_*(X) \).

A priori, depending on the choices of \( q \)-adapted subbundles \( E^- \subset E \), there are many possibilities for the d-manifold \( X_{dm} \) produced by this process. However, it is shown in [BJ17] that all such d-manifolds have the same bordism class and therefore must have the same virtual fundamental class. We record this in the following proposition.

**Proposition 2.9.** [BJ17, Corollary 3.19] The virtual fundamental class \( [X_{dm}]^{\text{vir}} \in H_*(X) \) associated to a proper, oriented \( (-2) \)-shifted symplectic derived scheme \( (\mathcal{X}, \omega_X) \) is independent of any choices involved in the construction of the d-manifold \( X_{dm} \) and thus canonical.

### 3. Normal Cones and Virtual Fundamental Classes

In this short section, we first briefly recall a topological version of the algebraic intrinsic normal cone [BF97] and its basic properties, following [Sie99, Section 3]. We then use the normal cone to give an alternative definition of the virtual fundamental class of a compact, oriented d-manifold \( \mathcal{X} \), which is equivalent to the one given in [Joy].

#### 3.1. Topological and homological normal cone

Let \( Y \) be an oriented, topological manifold of dimension \( N \) and \( E \) a topological vector bundle on \( Y \) with projection map \( q: E \to Y \). Let furthermore \( s \) be a continuous section of \( E \) with zero locus \( X = Z(s) \subset Y \).

For any \( \ell > 0 \), let \( p_\ell: Y \times \mathbb{R}^\ell \to Y \) be the projection and define a section \( s_\ell \in \Gamma(Y \times (\mathbb{R} \setminus \{0\}), p_\ell^* E) \) by the formula

\[
s_\ell(y, v) = |v|^{-1} \cdot s(y).
\]

Let \( \bar{\Gamma}_{s_\ell} \subset p_\ell^* E \) denote the closure of the graph of \( s_\ell \) in \( p_\ell^* E \).
Theorem-Definition 3.1. [Sie99, Definition 3.1, Proposition 3.2] The topological normal cone $C(s) \subset E$ associated to $s$ is defined as

\begin{equation}
C(s) = \Gamma_{s_\ell} \cap (Y \times \{0\}).
\end{equation}

This is independent of $\ell$ and satisfies:

1. $C(s) = \lim_{t \to \infty} \Gamma_{t \cdot s}$, where $C(s)$ and $\Gamma_{t \cdot s}$ are considered as closed subsets of $E$.
2. $C(s)$ lies over $X$ and $q(C(s)) = X$.

The long exact sequence in homology gives

\begin{equation}
H_{N+\ell}(C(s)) \to H_{N+\ell}(\Gamma_{s_\ell}) \to H_{N+\ell-1}(C(s))
\end{equation}

and therefore for large enough $\ell$ the two outer homology groups vanish and the homology class

\begin{equation}
(s_\ell)_* [Y \times (\mathbb{R}^\ell \setminus \{0\})] \in H_{N+\ell}(\Gamma_{s_\ell})
\end{equation}

gives rise to a fundamental class

\begin{equation}
[\Gamma_{s_\ell}] \in H_{N+\ell}(\Gamma_{s_\ell}).
\end{equation}

Remark 3.3. The homological normal cone should be regarded as the fundamental class of $C(s)$. This is motivated by the fact that formula (3.1) can obviously be equivalently written as $C(s_\ell) = \Gamma_{t \cdot s} \cap q^{-1}_t(0)$.

Finally, we remark that the topological normal cone is compatible with the intrinsic normal cone used in complex algebraic geometry. Suppose that $Y$ is a complex manifold, $E$ is a holomorphic vector bundle and $s$ a holomorphic section. Let $I$ be the ideal sheaf of $X$ in $Y$. The normal cone of $X$ associated to $s$ is then defined as

\begin{equation}
C_{X/Y} = \text{Spec}_{\mathcal{O}_X} \left( \oplus_{n \geq 0} I^n / I^{n+1} \right)
\end{equation}

and is a closed subcone of the vector bundle $E$. Here Spec is used to stand for MaxSpec, so that as a topological space the points of $C_{X/Y}$ correspond to maximal ideals.

Proposition 3.4. [Sie99, Proposition 3.5] $C_{X/Y} = C(s) \subset E$ and $[C_{X/Y}] = [C(s)] \in H_N(E)$. 

3.2. Virtual fundamental class. Let now $X$ be a compact, oriented $d$-manifold with underlying topological space $X$ and virtual dimension $n$.

By Theorem-Definition 2.2, $X$ is equivalent to a principal $d$-manifold, meaning that there exists an open subset $Y \subset \mathbb{R}^N$, a smooth vector bundle $E$ on $Y$ of rank $r$ and a smooth section $s$ of $E$ whose zero locus is $X$.

Moreover, we may assume that $Y$ retracts onto $X$ so that $f : H_i(X) \to H_i(Y)$ is an isomorphism for any $i$, where $f : X \to Y$ is the inclusion map.

The line bundle $L = \Lambda^r E \otimes \Lambda^N T^* Y$ is oriented. Since $\Lambda^N T^* Y$ admits an orientation by the definition of $Y$, this is equivalent to a choice of orientation of the bundle $E$. We may therefore further assume that $E$ is oriented.

By a small perturbation of the section $s$ of $E$, we get a section $s'$ of $E$ intersecting the zero section $Y$ transversely. Thus its zero locus $X'$ is a compact, oriented smooth manifold. Write $f' : X' \to Y$ for the inclusion.

**Definition 3.5.** [Joy] The J-virtual fundamental class of $X$ is defined as

$$[X]_{vir}^J = (f')^{-1} \circ f'_* [X'] \in H_n(X).$$

It is shown in [Joy] that the J-virtual fundamental class is independent of all choices involved and only depends on the bordism class of $X$.

Using the homological normal cone, we can also define a virtual fundamental cycle as follows.

**Definition 3.6.** The BF-virtual fundamental class of $X$ is defined as

$$[X]_{vir}^{BF} = 0_{E|X}^! [C(s)] \in H_n(X).$$

The next proposition shows that the two definitions are equivalent. In particular, the BF-virtual fundamental class is well defined and independent of any choices made above.

**Proposition 3.7.** $[X]_{vir}^J = [X]_{vir}^{BF}$.

**Proof.** By (3.4) and (3.5), we need to show

$$f'_* [X'] = f_* 0_{E|X}^! [C(s)] \in H_n(Y).$$

By the Cartesian diagram

$$\begin{array}{ccc}
E|_X & \xrightarrow{g} & E \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}$$

we have

$$f_* \circ 0_{E|X}^! = 0_{E}^! \circ g_*.$$  \hfill (3.7)

By Theorem-Definition 3.2, $g_* [C(s)] = [\Gamma_s] \in H_N(E)$ and for a small perturbation $s'$ of $s$ we have $[\Gamma_s] = [\Gamma_{s'}] \in H_N(E)$. Therefore

$$0_{E}^! g_* [C(s)] = 0_{E}^! [\Gamma_{s'}] = f'_* [X'] \in H_n(Y)$$

since $s'$ is taken to be transverse to the zero section of $E$.

Combining (3.6), (3.7) and (3.8) completes the proof.  \hfill $\square$
From now on, we write $[\mathcal{X}]^\text{vir}$ for the virtual fundamental class of the d-manifold $\mathcal{X}$.

**Remark 3.8.** Suppose that $Y$ is a complex manifold and $E$ and $s$ are holomorphic. Since the homological normal cone $[C(s)]$ coincides with the normal cone $[C_X/Y]$ by Proposition 3.4, it is clear by Definition 3.6 that the virtual fundamental class $[\mathcal{X}]^\text{vir}$ coincides with the definition of the virtual fundamental cycle in [BF97].

### 4. Vanishing for Surjective Cosections

Let $\mathcal{X}$ be a d-manifold whose truncation is the $C^\infty$-scheme $X_{\mathcal{C}}$. We typically denote the underlying topological space by $X$.

To simplify notation, we will write $R_X$ to denote both the trivial line bundle on $X_{\mathcal{C}}$ as an $O_{X_{\mathcal{C}}}$-module and the trivial real line bundle on the topological space $X$.

In this subsection, we prove a cone reduction criterion and vanishing theorems for the virtual fundamental classes of d-manifolds and ($-2$)-shifted symplectic schemes under appropriate notions of a surjective cosection.

#### 4.1. Cone reduction by real cosections

In this subsection, we verify that the arguments of [KL13, Section 4] go through in our context. We won’t need this discussion until we treat complex cosections in the next section, however it seems more appropriate to place it here.

Let $Y$ be a smooth manifold of dimension $N$, $E$ a smooth vector bundle on $Y$ and $s$ a smooth section of $E$ with zero locus $X = Z(s) \subset Y$. Locally, let $s_1, \ldots, s_r \in C^\infty(Y)$ be the components of the section $s$ and $y_1, \ldots, y_N$ coordinates on $Y$.

Now consider the principal d-manifold $\mathcal{X} = S_{Y,E,s}$ given by the above data. Write $p: E|_{X_{\mathcal{C}}} \to O_{\mathcal{X}}$ for the projection morphism. We have the following cone reduction statement (cf. [KL13, Corollary 4.5]).

**Proposition 4.1.** Let $\bar{\sigma}: O_{\mathcal{X}} \to R_X$ be a morphism of coherent sheaves on $X_{\mathcal{C}}$. Let $X(\sigma)$ be the locus where $\bar{\sigma} \circ p$ is zero and $U(\sigma) = X - X(\sigma)$ its complement, where $\bar{\sigma} \circ p$ is surjective. Then $C(s)$ is contained entirely in the kernel $F = \ker(\bar{\sigma} \circ p) = E|_{X(\sigma)} \cup \ker(\bar{\sigma} \circ p)|_{U(\sigma)}$.

**Proof.** Let $c = (x, e) \in C(s)$. We need to show that if $x \notin X(\sigma)$, then $c \in \ker(\bar{\sigma} \circ p)|_{U(\sigma)}$. As $X(\sigma)$ is closed in $X$, by shrinking around $x$ we may assume that $\sigma := \bar{\sigma} \circ p: E|_{X_{\mathcal{C}}} \to \mathbb{R}_X$ is surjective and extends to a smooth surjective map $E \to \mathbb{R}_Y$ on $Y$ and that we have fixed a connection $\nabla$ giving the tangent complex of $\mathcal{X}$, $ds: T_Y|_{X_{\mathcal{C}}} \to E|_{X_{\mathcal{C}}}$. We have by definition $\sigma|_{X_{\mathcal{C}}} \circ ds|_{X_{\mathcal{C}}} = 0$. This implies that for every variable $y_j$ of $Y$

\[
\sum_{i=1}^r \sigma_i \frac{\partial s_i}{\partial y_j} = O(s)
\]

where $O(s)$ stands for a linear combination of the functions $s_1, \ldots, s_r$ with coefficients smooth functions, and $\sigma_i$ are the components of $\sigma$. 

Fix \( \ell = 1 \) and \( c_1 = (x, 0, e) \in C(s) \subset p_1^* E \). By the definition of \( C(s) \), we have a sequence \((y_k, v_k) \in Y \times \mathbb{R}\) such that, as \( k \to \infty \), \( y_k \to x, v_k \to 0 \) and for \( i = 1, \ldots, r \)

\[
\lim_{k \to \infty} \frac{s_i(y_k)}{|v_k|} = e_i.
\]

Take any continuous path \( f(t) = (\gamma(t), \zeta(t)): [0, \epsilon) \to Y \times \mathbb{R} \) such that \( f(0) = (x, 0) \), \( f \) is differentiable on \((0, \epsilon)\), \( \zeta(t) \) is positive and strictly increasing on \((0, \epsilon)\) and there exists a sequence \( t_k \to 0 \) in \((0, \epsilon)\) such that \( f(t_k) = (y_k, v_k) \).

By re-parametrizing we may assume that \( \zeta(t) = t \) and \( t_k = v_k = |v_k| \).

(4.1) gives then for \( t \neq 0 \)

\[
\sum_{i=1}^{r} \sigma_i(\gamma(t)) \frac{\partial s_i}{\partial y_j}(\gamma(t)) \gamma_j'(t) = O(s(\gamma(t)))
\]

and hence by summing over \( j = 1, \ldots, N \)

\[
\sum_{i=1}^{r} \sigma_i(\gamma(t)) \frac{d}{dt} s_i(\gamma(t)) = O(s(\gamma(t))).
\]

It is immediate that

\[
\frac{d}{dt} \sum_{i=1}^{r} \sigma_i(\gamma(t)) s_i(\gamma(t)) = \sum_{i=1}^{r} a_i(t) s_i(\gamma(t))
\]

for some continuous functions \( a_i(t) \), smooth on \((0, \epsilon)\). We integrate to get

\[
\sum_{i=1}^{r} \sigma_i(\gamma(t)) s_i(\gamma(t)) = \int_{0}^{t} \sum_{i=1}^{r} a_i(\lambda) s_i(\gamma(\lambda)) d\lambda.
\]

Evaluating at \( t_k \) and dividing by \( t_k \) gives

\[
\sum_{i=1}^{r} \sigma_i(y_k) \frac{s_i(y_k)}{t_k} = \frac{1}{t_k} \int_{0}^{t_k} \sum_{i=1}^{r} a_i(\lambda) s_i(\gamma(\lambda)) d\lambda
\]

so letting \( k \to \infty \) it follows by (4.2) that

\[
\sigma(\epsilon) = \sum_{i=1}^{r} \sigma_i(x) e_i = \sum_{i=1}^{r} a_i(0) s_i(x) = 0
\]

which is what we want. \( \Box \)

4.2. Localization by surjective real cosections of \( d \)-manifolds. Let \( X \) be a \( d \)-manifold with underlying \( \mathcal{C}^\infty \)-scheme \( X_{\mathcal{C}^\infty} \). We give the following definition.

Definition 4.2. A real cosection is a morphism \( \sigma: \mathcal{O}_X \to \mathbb{R}_X \) of coherent sheaves on \( X_{\mathcal{C}^\infty} \).

A weak real cosection is a collection of linear maps \( \sigma_x: \mathcal{O}_X|_x \to \mathbb{R} \) for \( x \in X \) that vary continuously, i.e. for any open subset \( U \subset X \) and \( \mathcal{C}^\infty \)-vector bundle \( E_U \) on \( U \) with a surjection \( p_U: E_U \to \mathcal{O}_X|_U \) of sheaves on \( U_{\mathcal{C}^\infty} \) the linear maps \( \sigma_x \circ p_U|_x: E_U|_x \to \mathbb{R} \) define a continuous map of vector bundles \( E_U \to \mathbb{R}_U \).
A (weak) real cosection is surjective if the corresponding morphism(s) in its definition is (are) surjective.

**Remark 4.3.** Clearly a real cosection induces data of a weak real cosection.

We then have the following vanishing theorem.

**Theorem 4.4.** Let $\mathcal{X}$ be a compact, oriented $d$-manifold of virtual dimension $n$ with a surjective weak real cosection. Then

$$[\mathcal{X}]^{\text{vir}} = 0 \in H_n(\mathcal{X}),$$

i.e. the virtual fundamental class of $\mathcal{X}$ is zero.

**Proof.** By Theorem-Definition 2.2, $\mathcal{X}$ is equivalent to a principal $d$-manifold. Therefore, we have an open subset $Y \subset \mathbb{R}^N$, a smooth, oriented vector bundle $E$ on $Y$ of rank $r$ and a smooth section $s$ of $E$ with $X$ being its zero locus, which define an embedding of $C^\infty$-schemes $X_{C^\infty} \to Y$.

The tangent complex of $\mathcal{X}$ is given by

$$T_\mathcal{X} = [T_Y|_{X_{C^\infty}} \xrightarrow{ds} E|_{X_{C^\infty}}]$$

and hence we have a surjection $p: E|_{X_{C^\infty}} \to \mathcal{O}_X$. By the definition of weak cosection, the maps $\sigma_x \circ p_x: E|_x \to \mathbb{R}$ for $x \in X$ vary continuously and thus induce a continuous surjection of vector bundles

$$\sigma \circ p: E|_X \to \mathbb{R}_X.$$  

Since $[\mathcal{X}]^{\text{vir}} = 0|_X[C(s)]$, the virtual class is obtained by capping $[C(s)] \in H_N(E|_X)$ with the Euler class $e(E|_X) \in H^r_X(E|_X)$. But $E|_X$ has a trivial quotient (4.3), so $e(E|_X) = 0$, implying the vanishing of $[\mathcal{X}]^{\text{vir}}$. 

**Remark 4.5.** Theorem 4.4 can also be proved directly using Joyce’s definition of $[\mathcal{X}]^{\text{vir}}$ given in (3.4). We give the argument here, as it is quite short and similar to the above.

Keeping the same notation around (3.4), it is easy to see that $f_*[\mathcal{X}]^{\text{vir}} = e(E) = H_n(Y)$. After possibly shrinking $Y$ around $X$, since $X$ is compact, we may assume that the surjection (4.3) extends to a continuous surjection $E \to \mathbb{R}_Y$. Since $E$ has a trivial quotient, $e(E) = 0$ and we obtain $[\mathcal{X}]^{\text{vir}} = (f_*)^{-1}e(E) = 0 \in H_n(X)$.

**4.3. Cosection localization for $(-2)$-shifted symplectic derived schemes.**

We can give the following general definition of a cosection for any quasi-projective derived scheme $\mathcal{X}$.

**Definition 4.6.** A cosection is a morphism $\sigma: T_\mathcal{X}|_X[1] \to \mathcal{O}_X$ in $D^b(Coh \mathcal{X})$. We say that $\sigma$ is surjective if the morphism $h^0(\sigma): \mathcal{O}_Y \to \mathcal{O}_X$ is surjective, or equivalently if the collection of $\mathbb{C}$-linear maps $\sigma_x: \mathcal{O}_Y(\mathcal{X}, x) \to \mathbb{C}$ for every $x \in X$ are surjective.

Now suppose that $\omega_\mathcal{X}$ is a $(-2)$-shifted symplectic form on $\mathcal{X}$. We say that $\sigma$ is non-degenerate if the induced quadratic form $q_\sigma^\vee$ on $\mathcal{O}_Y(\mathcal{X}, x)^\vee$ is non-degenerate on the image of $\sigma_x^\vee$ for all $x \in X$.

**Remark 4.7.** When $\mathcal{X}$ is quasi-smooth so that $T_\mathcal{X}|_X$ is a perfect complex of amplitude $[0, 1]$, the above definition of a cosection coincides with the standard definition for schemes with perfect obstruction theory, as $h^1(T_\mathcal{X}|_X)$ is the obstruction sheaf of the perfect obstruction theory $\mathcal{L}_\mathcal{X}|_X \to \mathcal{L}_X$ on $X$. 

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We have the following theorem, saying that cosection localization applies to the setting of non-degenerate cosections, implying the vanishing of the virtual fundamental class.

**Theorem 4.8.** Suppose that a proper, oriented $(-2)$-shifted symplectic scheme $(\mathcal{X}, \omega_{\mathcal{X}})$ admits a non-degenerate cosection $\sigma$. Then its virtual fundamental class vanishes, i.e. $[\mathcal{X}_{dm}]^\text{vir} = 0 \in H_{\text{vir.dim.}}(X)$.

*Proof.* By Proposition 2.9, we are free to make any choice of $q$-adapted $E^- \subset E$ for an analytic Darboux chart $(V, E, G, q, s)$ around a point $x \in X$ in Construction 2.8.

Since $T_{\mathcal{X}|U}$ is isomorphic to the complex $G$, $\sigma$ is determined by a morphism $E \rightarrow \mathcal{O}_U$ and for every $u \in U$ we have the factorization

$$\sigma_u : \mathcal{O}(\mathcal{X}, u) \subset E|_u/\text{im}(ds|_u) \rightarrow \mathcal{O}_X|_u$$

and hence the projection

$$E|_u \rightarrow E|_u/\text{im}(ds|_u) \rightarrow \mathcal{O}_X|_u$$

is surjective. Therefore we get a surjective morphism $E|_U \rightarrow \mathcal{O}_U$, which up to possible shrinking of $V$ around $x$, extends to a surjective morphism $\sigma_E : E \rightarrow \mathcal{O}_V$.

The dual $\sigma_E^\vee : \mathcal{O}_V \rightarrow E^\vee$ restricts to $\sigma_u^\vee$ at $x$, as the differentials of the complex $G$ vanish at $x$ by construction. Thus, since $\sigma$ is non-degenerate, we may assume, up to further shrinking of $V$ around $x$, that the image of $\sigma_E^\vee$ is non-degenerate with respect to the quadratic form $q^\vee$ on $E^\vee$.

Taking the orthogonal complement of the image and dualizing, we get a $q$-orthogonal splitting $E = \mathcal{O}_V \oplus \ker(\sigma_E)$, such that $q$ is non-degenerate on each summand. After re-parametrizing analytically, $q$ is equal to $q_u(z) = y_1^2 + y_2^2 + \ldots + y_n^2$ on each fiber over $v \in V$, where $y_1$ is the coordinate on the fiber of the first trivial summand $\mathcal{O}_V$ and $y_2, \ldots, y_n$ are the fiber coordinates of the summand $\ker(\sigma_E)$.

We now take $E^-$ to be the direct sum of the imaginary parts of each individual summand. $E^-$ is easily seen to be $q$-adapted around $x \in X$ (it is obviously $q$-adapted at $x$ and being $q$-adapted is an open condition by [BJ17]).

Then, for $u \in X$ close to $x$, the subspace $\text{im} \Pi_u \subset \mathcal{O}(\mathcal{X}, u)$ must project onto $\mathbb{C}$ through $\sigma_u$ non-trivially since it is maximal negative definite with respect to $\text{Re} q_u$ as $E^-$ is $q$-adapted (see Definition 2.7). But, by the choice of $E^-$, $\text{im} \Pi_u$ consists only of imaginary vectors and therefore its projection to $\mathbb{C}$ is contained in $i\mathbb{R} \subset \mathbb{C}$, so we must have $\sigma_u(\text{im} \Pi_u) = i\mathbb{R}$. This implies that $\sigma_u$ induces a surjection

$$\sigma_u^+ : \mathcal{O}(\mathcal{X}, u)/\text{im} \Pi_u \rightarrow \mathbb{R}.$$  

For every such $u \in X$, the conditions in Definition 2.7 imply the existence of an exact sequence

$$T_{\mathcal{X}}|_u \xrightarrow{ds^+} E^+|_u \rightarrow \mathcal{O}(\mathcal{X}, u)/\text{im} \Pi_u \rightarrow 0.$$  

Since $\mathcal{X}_{dm}$ is locally equivalent to the principal d-manifold $S_{V,E^+,s^+}$, the maps $\sigma_u^+ : \mathcal{O}(\mathcal{X}_{dm})|_u \rightarrow \mathbb{R}$ vary smoothly and, in particular, give a surjective
weak real cosection for $X_{dm}$. Thus, by Theorem 4.4, we get the desired vanishing $[X_{dm}]^{\text{vir}} = 0 \in H_*(X)$. □

5. Localization by Complex Cosections

As usual, let $X$ be a compact, oriented $d$-manifold, $X_{C^\infty}$ the associated $C^\infty$-scheme and $X$ the underlying topological space. In this subsection, we study a certain kind of complex cosection that has similar properties to a cosection for an algebraic scheme with perfect obstruction theory.

Similarly to the case of real cosections, we write $C_{X}$ to denote both the trivial complex line bundle on $X_{C^\infty}$ and the trivial complex line bundle on $X$.

Definition 5.1. A complex cosection is a morphism $\sigma : \mathcal{O}_{X_{C^\infty}} \to C_{X}$ of sheaves on $X_{C^\infty}$.

From now on, we make the following holomorphicity assumptions regarding the topology of $X$ and the cosection $\sigma$.

Setup 5.2. We say that $X$ and $\sigma$ come from geometry if:

1. $X$ admits the structure of a complex analytic space $X^{an}$ inducing a structure of a $C^\infty$-scheme on $X$, denoted by $X_{C^\infty}^{an}$.
2. There exists a closed embedding $X^{an} \hookrightarrow W$, where $W$ is a complex manifold.
3. There exists a closed embedding of $C^\infty$-schemes $X_{C^\infty}^{an} \hookrightarrow X_{C^\infty}$.
4. The locus $X(\sigma)$ where $\sigma$ is not surjective, is a complex analytic closed subset of $X$.
5. The image of $\sigma|_{X_{C^\infty}^{an}}$ is an $\mathcal{O}_{X_{C^\infty}^{an}}$-submodule of $C_{X}$ generated by an ideal $I \subset \mathcal{O}_{X_{C^\infty}^{an}}$ whose zero locus is $X(\sigma)$.

We briefly clarify the above conditions.

In (1), if $X^{an}$ is analytically locally described as the vanishing locus of holomorphic functions $f_1, \ldots, f_r$ defined on a polydisc $D$ in $\mathbb{C}^N$, $X_{C^\infty}^{an}$ is locally described by the spectrum of the $C^\infty$-ring $C^\infty(D)/(\text{Re}f_1, \text{Im}f_1, \ldots, \text{Re}f_r, \text{Im}f_r)$.

Clearly homomorphic transition functions between such charts induce smooth transition functions so that we obtain a $C^\infty$-scheme.

In (5), if $I$ is locally generated by a set of holomorphic functions $f_1, \ldots, f_r$ on $X^{an}$ whose common zero set is $X(\sigma)$, we consider the $\mathcal{O}_{X_{C^\infty}^{an}}$-submodule of $\mathbb{C}_{X} = \mathbb{R}_{X} \oplus \mathbb{R}_{X}$ generated by the pairs

$$(\text{Re}f_1, \text{Im}f_1), (-\text{Im}f_1, \text{Re}f_1), \ldots, (\text{Re}f_r, \text{Im}f_r), (-\text{Im}f_r, \text{Re}f_r)$$

of functions on $X_{C^\infty}^{an}$. (5) requires the image of $\sigma|_{X_{C^\infty}^{an}}$ to be locally of this form for some choice of generators of $I$.

Therefore, Setup 5.2 essentially says that $X$ and the cosection $\sigma$ are determined by complex analytic data up to possible thickening of the underlying $C^\infty$-scheme.

Remark 5.3. If $X^{an}$ is a complex analytic space with an embedding $X^{an} \to W$ into a complex manifold and a perfect obstruction theory with holomorphic cosection $\sigma^{an}$, then the $d$-manifold $X$ obtained by applying the truncation functor in [Joy, Section 14.5], and its induced complex cosection $\sigma$ satisfy the conditions of Setup 5.2.
By Theorem-Definition 2.2, $\mathcal{X}$ is equivalent to a principal d-manifold $S_{Y,E,s}$. We fix a choice of such presentation
\begin{equation}
\mathcal{X} \simeq S_{Y,E,s}.
\end{equation}
Thus we have an embedding $X_{\mathbb{C}^\infty} \to Y$, given by the data of an open $Y \subset \mathbb{R}^N$, a smooth, oriented vector bundle $E$ on $Y$ of rank $r$, a smooth section $s$ of $E$ with $X$ being its zero locus and a surjection $p: E|_{X_{\mathbb{C}^\infty}} \to \mathcal{O}_X$.

Write $\sigma'$ for the composition $\sigma \circ p: E|_{X_{\mathbb{C}^\infty}} \to \mathbb{C}_X$ and $U(\sigma) = X - X(\sigma)$, the locus where $\sigma'$ is surjective. Let $E(\sigma) = E|_{X(\sigma)} \cup \ker (\sigma'|_{U(\sigma)})$.

Write $j: E|_{X(\sigma)} \to E(\sigma)$ for the inclusion.

In order to localize $[\mathcal{X}]_{\text{vir}}$, we need to use an appropriate notion of resolution of the cosection, as in [KL13] and [KL18].

**Definition 5.4.** We say that a proper map $\rho: \tilde{X} \to X$ of topological spaces is a $\sigma$-regularizing map with respect to the presentation (5.1) if:
\begin{enumerate}
\item $\rho|_{\rho^{-1}(U(\sigma))}: \rho^{-1}(U(\sigma)) \to U(\sigma)$ is a homeomorphism. Write $D$ for the complement of $U(\sigma)$ in $\tilde{X}$.
\item The pullback $\rho^*\sigma': \rho^*E \to \mathbb{C}_{\tilde{X}}$ factors through a surjection
\begin{equation}
\rho^*E|_X \to L
\end{equation}
where $L$ is a complex line bundle on $\tilde{X}$ together with a continuous section $t \in \Gamma(\tilde{X}, L^\vee)$ whose zero locus is $D$.
\end{enumerate}

We write $\rho(\sigma): D \to X(\sigma)$ for the induced map. Moreover let $E'$ be the kernel bundle of the surjection (5.2). We get an induced cartesian diagram of topological spaces
\begin{equation}
\begin{array}{ccc}
E' & \xrightarrow{\rho'} & \rho^*E|_X \\
\downarrow & & \downarrow \\
E(\sigma) & \xrightarrow{\rho} & E|_X.
\end{array}
\end{equation}

We say that a $\sigma$-regularizing map satisfies homological lifting if any homology class $\gamma \in H_i(E(\sigma))$ can be written as
\begin{equation}
\gamma = j_*\alpha + \rho'_*\beta
\end{equation}
for homology classes $\alpha \in H_i(E|_X(\sigma))$ and $\beta \in H_i(E')$.

The next proposition shows that when $\mathcal{X}$ and $\sigma$ come from geometry, there exists an induced $\sigma$-regularizing map that satisfies homological lifting.

**Proposition 5.5.** Suppose that the conditions in Setup 5.2 hold and we have fixed a presentation (5.1). Then there exists an induced $\sigma$-regularizing map $\rho: \tilde{X} \to X$ that satisfies homological lifting.

**Proof.** By parts (1) and (2) of Setup 5.2, we have a closed embedding $X^{an} \to W$. In particular, at the level of $\mathbb{C}^\infty$-schemes we get an embedding
\begin{equation}
X^{an}_{\mathbb{C}^\infty} \to W_{\mathbb{C}^\infty}.
\end{equation}
By definition, pulling back $E$ from $Y$ via the embedding $X_{\mathbb{C}^\infty} \to Y$ gives a vector bundle $E|_{X_{\mathbb{C}^\infty}}$ on $X_{\mathbb{C}^\infty}$. 

By part (3) of Setup 5.2, we can further pull it back to a vector bundle $E|_{X_{\text{an}}^\infty}$ on $X_{\text{an}}^\infty$. Pulling back $\sigma'$ gives a cosection
\begin{equation}
E|_{X_{\text{an}}^\infty} \rightarrow C_{X_{\text{an}}^\infty}.
\end{equation}

At the topological level, both vector bundles $E|_{X_{\infty}}$ and $E|_{X_{\text{an}}^\infty}$ give the vector bundle $E|_{X}$ on the underlying topological space $X$.

The existence of the embedding (5.5) and the bundle $E|_{X_{\text{an}}^\infty}$ implies (for example, by the argument in the proof of [Joy, Theorem 4.34]) that there exists an open neighbourhood $V$ of $X$ in $W$ such that $E|_{X}$ extends to a smooth vector bundle $E_{V}$ on $V$. Moreover, using parts (4) and (5) of Setup 5.2 and the cosection (5.6), up to shrinking $V$, we may extend the cosection $\sigma': E|_{X} \rightarrow C_{X}$ to a smooth morphism of vector bundles
\[\sigma_{V}: E_{V} \rightarrow C_{V}.\]

Let $\tau: \tilde{V} \rightarrow V$ be the blowup of $V$ along $X(\sigma)$ with exceptional divisor $D_{V}$ and set $\tilde{X} = \tilde{V} \times_{V} X$ with induced proper morphism $\rho: \tilde{X} \rightarrow X$. We then have a complex line bundle $L = O_{\tilde{V}}(-D_{V})|_{\tilde{X}}$ and a morphism
\[\tau^{*}\sigma_{V}: \tau^{*}E_{V} \rightarrow O_{\tilde{V}}(-D_{V})\]
whose restriction to $\tilde{X}$ gives a surjection
\[\rho^{*}\sigma': \rho^{*}E|_{X} \rightarrow L.\]

We thus see that $\rho: \tilde{X} \rightarrow X$ is $\sigma$-regularizing. It is easy to check that it is independent of the choices of $V$, $E_{V}$ and $\sigma_{V}$ and thus determined by the data provided by Setup 5.2.

Now, by construction, the diagram (5.3) gives rise to a Cartesian diagram
\begin{equation}
\begin{array}{ccc}
E' & \longrightarrow & \tau^{*}E_{V} \\
\rho'| & \downarrow & \downarrow \\
E(\sigma) & \longrightarrow & E_{V}.
\end{array}
\end{equation}
Therefore [KL18, Lemma 2.3] implies that $\rho$ satisfies homological lifting. \hfill \Box

**Remark 5.6.** The map $\rho$ is independent from the particular choice of closed embedding $X_{\text{an}} \rightarrow W$ and thus determined canonically by the rest of the data of Setup 5.2. What is essential is that such an embedding exists. This is true automatically for example when $X_{\text{an}}$ is quasi-projective.

Given a $\sigma$-regularizing map $\rho: \tilde{X} \rightarrow X$ satisfying homological lifting, by [KL18] we have a localized Gysin map
\begin{equation}
0^{l}_{E|_{X},\sigma}: H_{*}(E(\sigma)) \longrightarrow H_{*-r}(X(\sigma))
\end{equation}
given by the formula
\begin{equation}
0^{l}_{E|_{X},\sigma}\gamma = 0^{l}_{E|_{X(\sigma)}}\alpha - \rho(\sigma)*\left(\epsilon(L^{\vee}, t) \cap 0^{l}_{E'}\beta\right)
\end{equation}
where $\gamma \in H_{i}(E(\sigma))$ and we have written $\gamma = j_{*}\alpha + \rho'_{*}\beta$ for some $\alpha \in H_{i}(E|_{X(\sigma)})$ and $\beta \in H_{i}(E')$.

By the proof of [KL18, Theorem 3.2], the Gysin map $0^{l}_{E|_{X},\sigma}$ is independent from the particular choice of $\alpha$ and $\beta$. 

If \( i : X(\sigma) \to X \) denotes the inclusion, it satisfies
\[
0^i_{E|X} \circ j_s = i_* \circ 0^i_{E|X,\sigma} : H_*(E|X) \to H_{*-r}(X).
\]

Now, by Proposition 4.1, \( C(s) \) is contained in \( E(\sigma) \) and therefore by the definition (3.3) it follows that we obtain a class
\[
[C(s)] \in H_n(E(\sigma)).
\]

We are now in position to define the cosection localized virtual fundamental class of \( X \).

**Theorem-Definition 5.7.** Let \( X \) be a compact, oriented d-manifold of virtual dimension \( n \) and \( \sigma : \mathcal{O}_Y X \to \mathcal{C}_X \) a complex cosection, satisfying the conditions of Setup 5.2. Then the cosection localized virtual fundamental class of \( X \) is defined by
\[
[X]^{\text{vir}}_{\text{loc}, \sigma} := 0^{i}_{X(\sigma)} [C(s)] \in H_n(X(\sigma)).
\]

It satisfies
\[
i_* [X]^{\text{vir}}_{\text{loc}, \sigma} = [X]^{\text{vir}} \in H_n(X).
\]

Finally, we have not addressed the potential dependence of \( [X]^{\text{vir}}_{\text{loc}, \sigma} \) on the presentation (5.1) of \( X \) as a principal d-manifold, i.e. on the data \( Y, E, s \).

**Proposition 5.8.** \( [X]^{\text{vir}}_{\text{loc}, \sigma} \) is independent from the choice of presentation \( X \simeq \mathcal{S}_{Y,E,s} \) and thus well defined.

**Proof.** It is clear that the \( \sigma \)-regularizing map \( \rho : \tilde{X} \to X \) is independent from the particular choice of presentation.

Suppose now that \( X \simeq \mathcal{S}_{Y_1,E_1,s_1} \simeq \mathcal{S}_{Y_2,E_2,s_2} \). By [Joy, Theorems 1.4.8, 4.9], up to possibly shrinking \( Y_1 \) around \( X \), we can write this equivalence in the following standard form: we have a submersion \( f : Y_1 \to Y_2 \) and a morphism of vector bundles \( \hat{f} : E_1 \to f^*E_2 \) such that \( \hat{f} \circ s_1 = f^*s_2 \). The morphism of tangent complexes
\[
\begin{array}{ccc}
T_{Y_1}|_X & \xrightarrow{\text{d} s_1} & E_1|_X \\
\text{d} f \downarrow & & \downarrow \hat{f} \\
\text{d} f^*T_{Y_2}|_X & \xrightarrow{\text{d} (f^*s_2)} & f^*E_2|_X
\end{array}
\]
is a quasi-isomorphism on fibers over \( x \in X \) and thus we must have that \( E_1|_X \xrightarrow{\hat{f}} f^*E_2|_X \) is surjective, so after further possible shrinking of \( Y_1 \), we may assume that \( \hat{f} \) is surjective.
Using the notation of diagram 5.3, we have an induced diagram with Cartesian squares

$$
\begin{array}{ccc}
E'_1 & \xrightarrow{\tilde{f}} & (f^*E_2)' \\
\downarrow \rho'_1 & & \downarrow \rho'_2 \\
E_1(\sigma) & \xrightarrow{\tilde{f}(\sigma)} & (f^*E_2)(\sigma) \\
\downarrow i_1 & & \downarrow i_2 \\
E_1|_{X(\sigma)} & \xrightarrow{\tilde{f}|_{X(\sigma)}} (f^*E_2)|_{X(\sigma)} & \xrightarrow{f|_{X(\sigma)}} E_2|_{X(\sigma)}
\end{array}
$$

where the horizontal arrows are smooth and the vertical arrows proper.

Write $\tilde{g}' = f' \circ \tilde{f}$, $g(\sigma) = f(\sigma) \circ \tilde{f}(\sigma)$ and $g|_{X(\sigma)} = f|_{X(\sigma)} \circ \tilde{f}|_{X(\sigma)}$ for the (smooth) compositions of the horizontal arrows.

The equivalence $S_{Y_1,E_1,s_1} \simeq S_{Y_2,E_2,s_2}$ and the definition of the cones $[C(s_1)]$ and $[C(s_2)]$ imply that

$$
[C(s_1)] = \tilde{f}(\sigma)^* [C(f^*s_2)] = \tilde{f}(\sigma)^* f(\sigma)^* [C(s_2)] = g(\sigma)^* [C(s_2)]
$$

and therefore if

$$
[C(s_2)] = (i_2)_*\alpha_2 + (\rho'_2)_*\beta_2
$$

in the notation of (5.4), we will have

$$
[C(s_1)] = (i_1)_*\alpha_1 + (\rho'_1)_*\beta_1
$$

where

$$
\alpha_1 = (g|_{X(\sigma)})^* \alpha_2, \quad \beta_1 = (\tilde{g}')^* \beta_2.
$$

By the usual properties of Gysin maps (cf. for example [KL18]), we obtain

$$
0^I_{E_1|_{X(\sigma)}} \alpha_1 = 0^I_{E_2|_{X(\sigma)}} \alpha_2, \quad 0^I_{E_1} \beta_1 = 0^I_{E_2} \beta_2
$$

so the formula of the localized Gysin map (5.9) immediately implies that

$$
0^I_{E_1|_{X(\sigma)}} [C(s_1)] = 0^I_{E_2|_{X(\sigma)}} [C(s_2)]
$$

as we want. □

**Remark 5.9.** In general, given a cosection $\sigma$ on $X$, it is not clear if it is possible to endow the locus $X(\sigma)$ with a natural choice of structure of a d-manifold.

This is intuitive from the point of view of complex analytic spaces with perfect obstruction theory, as a cosection does not necessarily induce a perfect obstruction theory on its vanishing locus.

### 6. Application: Vanishing of Stable Pair Invariants of Hyperkähler Fourfolds

Let $W$ be a smooth, projective Calabi-Yau fourfold. In [CMT19], the authors study the stable pair invariants of $W$ and claim that they vanish when $W$ is hyperkähler, anticipating the existence of a cosection localization theory for d-manifolds. In this section, we prove their claim using the theory developed in the previous parts of the paper.
A stable pair on $W$ is the data of 

$$(F, s), ~ F \in \text{Coh } W, ~ s: \mathcal{O}_W \longrightarrow F$$

where $F$ is a pure one-dimensional sheaf on $W$ and $s$ is surjective in dimension one.

Fix $\beta \in H_2(W, \mathbb{Z})$ and $n \in \mathbb{Z}$. Then the moduli space $P_n(W, \beta)$ is a projective scheme which parameterizes stable pairs viewed as two-term complexes

$$I = [\mathcal{O}_W \xrightarrow{s} F] \in D^b(\text{Coh } W)$$

satisfying $[F] = \beta$ and $\chi(F) = n$.

By [CMT19, Lemma 1.3], $X = P_n(W, \beta)$ is the truncation of a $(-2)$-shifted symplectic derived scheme $\mathcal{X}$, which admits a choice of orientation. Writing $\mathbb{I} = [\mathcal{O}_W \times X \rightarrow \mathbb{F}]$ for the universal stable pair and $\pi_X: X \times W \rightarrow X, \pi_W: X \times W \rightarrow W$ for the projection maps, we have

$$\mathbb{T}_{\mathcal{X}}|_X = R\text{Hom}_{\pi_X}(I, I)_0[1]$$

so that at a point $x = [I] \in X$ we have $\text{Ob}(\mathcal{X}, x) = \text{Ext}^2(I, I)_0$ and the quadratic form $q_x = q_I$ induced by the symplectic form coincides with the Serre duality pairing.

Suppose now that $W$ is hyperkähler with holomorphic symplectic form $\eta$. Then, using the Atiyah class of $I$ and the symplectic form, it is shown in [CMT19] that $\mathcal{X}$ admits a strong cosection, whose associated weak cosection is non-degenerate.

We briefly recall the construction for the convenience of the reader. Let

$$\text{At}(I) \in \text{Ext}^1(\mathbb{I}, \mathbb{I} \otimes \pi_W^* \Omega_W)$$

be the relative Atiyah class of $I$. We then get morphisms

$$\wedge \frac{\text{At}(\mathbb{I})^2}{2}: \mathbb{T}_{\mathcal{X}}|_X[1] = R\text{Hom}_{\pi_X}(I, I)_0[2] \longrightarrow R\text{Hom}_{\pi_X}(I, I \otimes \pi_W^* \Omega^2_W)[2]$$

$$\cup \eta: R\text{Hom}_{\pi_X}(I, I \otimes \pi_W^* \Omega^2_W)[2] \longrightarrow R\text{Hom}_{\pi_X}(I, I[2])[2]$$

$$\text{tr}: R\text{Hom}_{\pi_X}(I, I[2])[2] \longrightarrow R\text{Hom}_{\pi_X}(\mathcal{O}_{X \times W}, \mathcal{O}_{X \times W}[2])[2] = \mathcal{O}_X \otimes \mathcal{O}_X[-4]$$

Projecting onto $\mathcal{O}_X$, their composition gives a cosection

$$\sigma: \mathbb{T}_{\mathcal{X}}|_X[1] \longrightarrow \mathcal{O}_X$$

which at the point $x = [I] \in X$ corresponds to a morphism

$$(6.1) \quad \sigma_x = \sigma_I: \text{Ob}(\mathcal{X}, x) = \text{Ext}^2(I, I)_0 \longrightarrow H^4(W, \mathcal{O}_W).$$

We have the following proposition.

**Proposition 6.1.** [CMT18, Proposition 2.9] [CMT19, Proposition 2.18] Let $W$ be a smooth, projective hyperkähler fourfold and $X$ a connected scheme. Let $\mathbb{I}$ be any perfect complex on $X \times W$ and write $\mathbb{I}|_x = I$ for $x \in X$. Then if $\text{ch}_3(I) \neq 0$ or $\text{ch}_4(I) \neq 0$, $\sigma_I$ is surjective and:

1. If $\text{ch}_4(I) \neq 0$, $\sigma_I$ admits a $q_I$-orthogonal splitting and $q_I$ is non-degenerate on $H^4(W, \mathcal{O}_W)$. 


(2) If \( \text{ch}_4(I) = 0 \) and \( \text{ch}_3(I) \neq 0 \), let \( \kappa_W \in H^1(W, T_W) \) be the Kodaira-Spencer class dual to \( \text{ch}_3(I) \). Similarly to the above, we obtain a second strong cosection
\[
\tau : \pi_W^* \kappa_W \circ (\wedge \text{At}(I)) : T_X[x][1] \to \mathcal{O}_X.
\]
The map \( \sigma_I \oplus \tau_I \) is surjective and admits a \( q_I \)-orthogonal splitting such that \( q_I \) is non-degenerate on \( H^4(W, \mathcal{O}_W) \). In particular,
\[
(6.2) \quad \sigma_I + \tau_I : \text{Ext}^2(I, I)_0 \to H^4(W, \mathcal{O}_W)
\]
admits a \( q_I \)-orthogonal splitting and \( q_I \) is non-degenerate on \( H^4(W, \mathcal{O}_W) \).

Using the proposition, it is immediate by Definition 4.6 that if \( n \neq 0 \), the cosection \( \sigma \) for \( P_n(W, \beta) \) is non-degenerate, and if \( \beta \neq 0 \), the cosection \( \sigma + \tau \) given pointwise by (6.2) is non-degenerate.

Therefore by Theorem 4.8 we immediately get the following vanishing result for stable pair invariants on hyperkähler fourfolds, which is precisely [CMT19, Claim 2.19].

**Theorem 6.2.** Let \( W \) be a smooth, projective hyperkähler fourfold and \( P_n(W, \beta) \) the moduli space of stable pairs with \( n \neq 0 \) or \( \beta \neq 0 \). Then \([P_n(W, \beta)]^{\text{vir}} = 0\).

**References**


