Cohomology of Coherent Sheaves on Complex Algebraic Varieties and Hodge Theory

Michail Savvas

May 2012

Contents

1 Introduction

In this essay we will be concerned with complex algebraic varieties. A particularly nice thing about these objects is that they can be viewed in two ways: we can examine their properties and behaviour from a strictly algebraic point of view, i.e. through the lens of algebraic geometry, and also from an analytic point of view since they can be endowed with the structure of a complex analytic space (in particular, of a complex manifold if non-singular), which gives us the freedom to apply results from complex manifold theory and analysis.

Our main focus in this exposition will be on the one hand to examine the interplay between these two approaches and on the other to present some results about the cohomology of complex algebraic varieties. More precisely, we will see when the application of algebraic techniques, which are in general more rigid and powerful in this setting, can successfully substitute topological or analytic methods. It will be the case that many statements about the singular/de Rham or sheaf cohomology can often be reduced under certain assumptions to algebraic statements and computations (Section 3 - GAGA Theorem, Section 4 - Grothendieck's algebraic de Rham Theorem, Section 5 - degeneracy of the Hodge to de Rham spectral sequence). Moreover, we will discuss the Hodge Theorem and Hodge decomposition for the cohomology of a compact, Kähler manifold and thus for a smooth projective complex variety.

We begin with stating some known facts in Section 2, which serves an introductory purpose. These regard coherent sheaves on algebraic varieties and cohomology of coherent sheaves. The results mentioned can be found in [2], [11, Chapter III], [12] and [13].

Section 3 is about Jean-Pierre Serre's GAGA Theorem. Our treatment follows closely Serre's original paper [1]. We see that any complex algebraic variety X has the structure of a complex analytic space X^{an} in a natural way. This extends to a correspondence between coherent algebraic sheaves on X and coherent analytic sheaves on X^{an}. GAGA shows that in the case of projective varieties there is an equivalence between these two categories and also that sheaf cohomology stays invariant under this correspondence. We present Serre's original proof with few omissions and in addition some neat applications, e.g. Chow's theorem.

In Section 4 we give an exposition of Grothendieck's algebraic de Rham theorem. It draws mainly from Grothendieck's original paper [3] together with some input from [8], [9] and [14]. We first define the algebraic de Rham cohomology of a smooth complex algebraic variety using hypercohomology. Grothendieck's Theorem says that this coincides with the analytic de Rham cohomology. Hence, for smooth complex varieties, the algebraic de Rham, analytic de Rham and complex singular cohomology all coincide.

Finally, Section 5 deals with the Hodge theory of compact, Kähler manifolds and thus, if the reader prefers, smooth projective complex varieties in particular, using input from [8], [9] and [4], [5], [6], [7]. We see how the use of results from analysis leads to the Hodge Theorem and Hodge decomposition for the cohomology of compact, Kähler manifolds. Then we will demonstrate in which ways we can utilise algebraic methods along with results of the two preceding sections to approximate these theorems. In this context, we define Hodge structures, the Hodge filtration and the Hodge to de Rham spectral sequence and exhibit some of their properties. Finally, we conclude the essay by briefly presenting some nice results concerning the behaviour of the Hodge decomposition under small deformations of the complex structure on a smooth manifold.

Throughout, we assume basic results from algebraic geometry, complex geometry and homological algebra. We choose though to state basic definitions and lemmas, when they are considered absolutely central or appear often, as we feel that this approach makes the text more self-contained and is honest to the reader. This justifies the existence of Section 2, along with some introductory/foundational subsections in the next sections. For more details or any ommissions, the reader can look at [2], [11], [12], [8] or [9]. We also assume that the notation we have used is reasonably standard and thus choose not to give relevant explanations, unless a new notion is being introduced.

Acknowledgements I would like to deeply thank Prof. Ian Grojnowski for supervising me in this essay and introducing me to these really nice pieces of mathematics. He has been extremely helpful and willing to meet and discuss my questions throughout. I am really grateful for all our interaction, which led me to study and understand these amazing ideas, and his support in general.

2 Coherent Sheaves and their Cohomology

In this short section we will mention several definitions and propositions without proofs which will be used in the rest of the essay. These will range from very basic and easy to show to quite hard. We will give references throughout. As one might expect, contrary to its title maybe, this section is not quite coherent.

2.1 Coherent sheaves: Basic definitions, properties and operations

It is probably unjust to omit the definition of a coherent sheaf and this is how we begin.

Definition 2.1.1. (Coherent Sheaf) Let $(X, \mathcal{O}_X)$ be a ringed space. Then an $\mathcal{O}_X$-module $\mathcal{F}$ will be called a **coherent sheaf** on X if locally it admits a presentation

$$\mathcal{O}_X^p \longrightarrow \mathcal{O}_X^q \longrightarrow \mathcal{F} \longrightarrow 0$$

where $p,\ q$ are non-negative integers and the sequence is exact. Here $\mathcal{O}_X^p$ stands for the direct sum of p copies of $\mathcal{O}_X$.

We also say that a sheaf of rings $\mathcal{F}$ on X is **coherent** if it is coherent when considered as an $\mathcal{F}$-module (i.e. if it satisfies the above definition for the ringed space $(X, \mathcal{F})$).

In what follows in this section, all sheaves will be $\mathcal{O}_X$-modules for a ringed space $(X, \mathcal{O}_X)$. Set $\mathcal{O} = \mathcal{O}_X$ for brevity. The next proposition gives an account of the main properties of coherent sheaves.

Proposition 2.1.2. 1. Let $0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0$ be an exact sequence of sheaves. Then if two of them are coherent, so is the third.
2. If $\phi : \mathcal{F} \longrightarrow \mathcal{G}$ is a morphism between coherent sheaves, then the kernel, image and cokernel ϕ are coherent.

Proof See [2, no 13, Theorems 1 & 2]. □

As a corollary of this proposition, we may obtain further properties, given in the following proposition, which treats some sheaf operations.

Proposition 2.1.3. 1. A direct sum of coherent sheaves is coherent.
2. If $\mathcal{F}, \mathcal{G}$ are coherent, so is the sheaf $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$.
3. If $\mathcal{F}$ is coherent and $\mathcal{G}$ any sheaf, then for all $x \in X$, $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})_{\mathrm{x}}$ is isomorphic to $\mathrm{Hom}_{\mathcal{O}_{\mathrm{x}}}(\mathcal{F}_{\mathrm{x}}, \mathcal{G}_{\mathrm{x}})$.
4. If $\mathcal{F}, \mathcal{G}$ are coherent, so is the sheaf $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$.

Proof See [2, no 13 & 14]. □

We now state a fact about coherent sheaves on projective varieties, which will be of use later on.

Lemma 2.1.4. Let X a complex projective variety and $\mathcal{F}$ a coherent sheaf on X. Then there exist integers $p \geq 0$ and $n \in \mathbb{Z}$ and a surjection $\mathcal{O}_X(n)^p \longrightarrow \mathcal{F} \longrightarrow 0$.

The proof of the Lemma relies on the known fact that coherent sheaves on projective varieties have a particularly nice structure. For a more detailed discussion, the interested reader can look at [12, pp. 56-62].

Examples 2.1.5.
1. Trivially for any ringed space $(X, \mathcal{O}_X)$, the structure sheaf $\mathcal{O}_X$ is coherent.
2. **(The sheaves $\mathcal{O}(n)$)** Let $X = \mathbb{P}^r(\mathbb{C})$, $\mathcal{O} = \mathcal{O}_X$ its structure sheaf and $n \in \mathbb{Z}$. Let also $\mathcal{U} = \{U_i\}$ be the standard open cover of X by $r+1$ affine sets, $\mathcal{O}_i = \mathcal{O}|_{U_i}$ and $\sigma_{ij} : \mathcal{O}_i|_{U_i \cap U_j} \longrightarrow \mathcal{O}_j|_{U_i \cap U_j}$ the sheaf morphism given by multiplication by X_j^n / X_i^n. Then the maps σ_{ij} are isomorphisms and moreover they satisfy the cocycle condition $\sigma_{ij} \circ \sigma_{jk} = \sigma_{ik}$ over $U_i \cap U_j \cap U_k$. So we may glue the sheaves $\mathcal{O}_i$ with respect to these isomorphisms to obtain a new sheaf $\mathcal{O}(n)$. Since locally $\mathcal{O}(n)$ and $\mathcal{O}$ are the same, it follows that $\mathcal{O}(n)$ is coherent.
These sheaves admit also an alternative description: Let $\pi : \mathbb{C}^{r+1} \setminus 0 \longrightarrow \mathbb{P}^r(\mathbb{C})$ be the natural projection. Then we may define $\mathcal{O}(n)$ by $\mathcal{O}(n)(U) = \{$homogeneous regular functions on $\pi^{-1}(U)$ of degree $n\}$ for every open subset U of $\mathbb{P}^r(\mathbb{C})$.
Finally, it is easy to see that these sheaves are invertible and correspond thus to line bundles on projective space. We will later see what their cohomology is and they will come up in many of the proofs, which is why we give their explicit definition.
3. **(The sheaves $\mathcal{F}(n)$)** Let $X \subset \mathbb{P}^r(\mathbb{C})$ a projective variety and $\mathcal{O}_X$ its structure sheaf. Denote $\mathcal{O}_X(n) = \mathcal{O}_{\mathbb{P}^r(\mathbb{C})}(n)|_X$. Then if $\mathcal{F}$ is a sheaf on X, we define the sheaf $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$. By the above Propositions, we see that if $\mathcal{F}$ is coherent, so is $\mathcal{F}(n)$.

2.2 Cohomology of projective space

Here we state two results without proof about the cohomology of complex projective space $X = \mathbb{P}^r(\mathbb{C})$ with coefficients in the line bundles $\mathcal{O}(n)$ (see Examples 2.1.5.) and sheafs of p-forms Ω_X^p.

Lemma 2.2.1. 1. $H^i(X, \mathcal{O}(n)) = 0$ unless $i = 0$ or r.
2. $H^0(X, \mathcal{O}(n)) = \{f \in \mathbb{C}[X_0, ..., X_r] : f$ is homogeneous of degree $n\}$.
3. $H^r(X, \mathcal{O}(n)) = \{f \in \mathbb{C}[X_0^{-1}, ..., X_r^{-1}]X_0^{-1} ... X_r^{-1} : f$ is homogeneous of degree $n\}$.

Lemma 2.2.2. $H^q(X, \Omega_X^p) = 0$ if $p \neq q$ and otherwise $H^q(X, \Omega_X^p) = \mathbb{C}$.

We assume throughout that the reader is familiar with all the definitions related to sheaf cohomology, both in terms of derived functors and the more concrete setup of Čech cohomology, as well as with Leray's theorem and how it can be applied to algebraic varieties. For details and the actual computations of the above, we refer the interested reader to [11, Chapter III] and [12].

2.3 Finite dimensionality of cohomology groups and vanishing theorems

Theorem 2.3.1. Let X be a projective variety and $\mathcal{F}$ a coherent sheaf on X. Then:
1. The cohomology groups $H^i(X, \mathcal{F})$ are finite dimensional as complex vector spaces.
2. There exists $n_0 \in \mathbb{Z}$ such that $H^i(X, \mathcal{F}(n)) = 0$ for $i > 0$ and $n \geq n_0$.

Theorem 2.3.2. (Serre's vanishing principle) Let X be an affine variety and $\mathcal{F}$ a coherent sheaf on X. Then $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.

Theorem 2.3.3. (Special case of Grothendieck's vanishing theorem) Let X be a variety of dimension d and $\mathcal{F}$ a sheaf of abelian groups on X. Then $H^i(X, \mathcal{F}) = 0$ for all $i > d$.

For more details and more general/complete statements of the above theorems and proofs, the reader may look at [11, Chapter III].

3 Serre's GAGA

3.1 Introduction

In this section, we will present the basic results in Jean-Pierre Serre's famous paper "Géométrie Algébrique et Géométrie Analytique", commonly abbreviated as GAGA (we have been and will be using this abbreviation throughout this whole essay). The essence of the basic theorem first proved in GAGA is that it gives a strong connection between analytic and algebraic properties of complex projective varieties and shows that to a certain extent the two viewpoints are equivalent and yield the same results. We will refer to this interchangeably as the GAGA Theorem/Principle or just GAGA.

We will mainly follow the exposition in Serre's paper, exhibiting and proving the absolutely necessary results and lemmas that lead to the main theorem and give the corresponding references, whenever a proof is omitted. We will also present some applications of GAGA (for this see also in the next chapters), as well as some neat results that give relations between analytic and algebraic properties of complex varieties and suit this general framework.

3.2 The basics: Definitions, notions and properties

In this section, we will give the basic definitions that we will need in what follows. We will also state essential properties, some of which we will also prove. While many will be very basic, we still state them for the sake of completeness and self-containment, at least at a conceptual level (when not technical).

For the definition of an algebraic variety, we stick to the one given by Serre and thus refer the reader to [2, no 34]. However, it is not difficult to see that everything which follows still goes through in more generality for schemes of finite type over $\mathbb{C}$.

Definition 3.2.1. (Analytic set) We say that a subset U of an affine space $\mathbb{C}^n$ is **analytic** if it is locally given by the vanishing of finitely many holomorphic functions, i.e. if for every $x \in U$ there exists a neighbourhood W of x and functions $f_1, ..., f_k$ holomorphic on W such that $U \cap W = \{z \in W \ : \ f_1(z) = ... = f_k(z) = 0\}$.

It is clear that U comes equipped with a natural sheaf $\mathcal{H}_U$ of germs of holomorphic functions on it, if we view it as a subspace of $\mathbb{C}^n$. More precisely, let $\mathcal{A}(U)$ denote the sheaf of germs of holomorphic functions on $\mathbb{C}^n$ whose sections over an open W are exactly those functions which are zero on $W \cap U$. This is a sheaf of ideals of the sheaf $\mathcal{H}$ of germs of holomorphic functions on $\mathbb{C}^n$. Then $\mathcal{H}_U$ is the quotient sheaf $\mathcal{H}/\mathcal{A}(U)$. So we see that holomorphic functions on U come from restricting holomorphing functions on $\mathbb{C}^n$ with the obvious identification (when restrictions are equal), as one would expect.

Definition 3.2.2. (Holomorphic map between analytic sets) Let $U \subseteq \mathbb{C}^r$, $V \subseteq \mathbb{C}^s$ be two analytic sets. Then a map $f : U \longrightarrow V$ is **holomorphic** if it is continuous and transforms the sheaf $\mathcal{H}_\mathcal{V}$ into the sheaf $\mathcal{H}_\mathcal{U}$, i.e. for any $g \in \mathcal{H}_{V,f(x)}$ we have $g \circ f \in \mathcal{H}_{U,x}$.

Definition 3.2.3. (Analytic isomorphism) Let $U \subseteq \mathbb{C}^r$, $V \subseteq \mathbb{C}^s$ be two analytic sets. A map $f : U \longrightarrow V$ is an **analytic isomorphism** if it is holomorphic and admits a holomorphic inverse f^{-1}.

Having given these definitions, we proceed to generalise them slightly.

Definition 3.2.4. (Analytic space) Let X be a topological space together with a subsheaf $\mathcal{H}_X$ of the sheaf on X of germs of complex-valued functions. We will say that $(X, \mathcal{H}_X)$ is an **analytic space** if the following hold:
1. X is locally an analytic set, i.e. for every $x \in X$ there exists a neighbourhood U of x such that W with the topology and the sheaf induced by those on X is isomorphic to an analytic set V with the topology and sheaf defined as above. This isomorphism is understood as an isomorphism between locally ringed spaces.
2. X is Hausdorff.

Definition 3.2.5. (Analytic sheaf) Let $(X, \mathcal{H}_X)$ be an analytic space. An **analytic sheaf** $\mathcal{F}$ on X is just a sheaf of $\mathcal{H}_X$-modules.

Now we may give analogous definitions as above in the obvious way in order to make sense of holomorphic maps and isomorphisms between analytic spaces, which we omit. Also, if Y is a closed analytic subspace of an analytic space X, we may define the analytic sheaf $\mathcal{A}(Y)$ as in the above and identify $\mathcal{H}_Y$ with the quotient $\mathcal{H}_X/\mathcal{A}(Y)$ as before. The sheaves $\mathcal{H}_X$ and $\mathcal{A}(Y)$ are coherent, as shown in the next Lemma.

Lemma 3.2.6. Let $(X, \mathcal{H}_X)$ be an analytic space and $Y \subseteq X$ a closed analytic subspace. Then $\mathcal{H}_X$ is a coherent sheaf of rings and $\mathcal{A}(Y)$ is a coherent analytic sheaf.

Proof The proof of this Lemma makes use of the
Oka Coherence Theorem: If $U \subseteq \mathbb{C}^n$ is open, then $\mathcal{H}_U$ is a coherent sheaf of rings.
This together with a result of H. Cartan establish the Lemma when X is an open subset of $\mathbb{C}^n$. The local nature of the statement allows us to assume that X is a closed analytic subset of an open set $U \subseteq \mathbb{C}^n$. Then by our earlier identification $\mathcal{H}_X = \mathcal{H}_U/\mathcal{A}(X)$ and the above we get that $\mathcal{H}_U$ is a coherent sheaf of rings and $\mathcal{A}(X)$ is a coherent sheaf of ideals of $\mathcal{H}_U$. Hence $\mathcal{H}_X$ is coherent, using elementary properties of coherent sheaves (see [2] for more detail). The proof of the statement about $\mathcal{A}(Y)$ is similar. □

Remark We see that the proof given above is essentially a sketch - reduction to the case of an open subset of $\mathbb{C}^n$. This is the significant part of the proof, which we take for granted assuming the two theorems by Oka and Cartan.

Having gone through this chain of definitions and seen that the sheaves we will be interested in are coherent, we proceed to show how we can associate an analytic space X^{an} to an algebraic variety X over the complex numbers and also an analytic sheaf $\mathcal{F}^{an}$ on X^{an} to an algebraic sheaf $\mathcal{F}$ on X.

Construction of X^{an} Let $\mathcal{U} = \{U_i\}$ be a cover of X by affine open subsets, that is every U_i is isomorphic to a Zariski closed subset V_i of some affine space $\mathbb{C}^n$. Since polynomials are holomorphic functions, we see that V_i is an analytic set and hence we may transport the analytic structure of V_i to U_i via these isomorphisms by requiring they become analytic isomorphisms. In this way, it is clear (and trivial to check the Hausdorffness axiom) that we can endow X with the structure of an analytic space $(X^{an}, \mathcal{H}_{X^{an}})$.

Remarks 1. We see by the construction that the analytic topology on X^{an} is finer than the Zariski topology on X, i.e. the map $id_X : X^{an} \longrightarrow X$ is continuous, and is also the least fine analytic topology such that all regular functions on Zariski open subsets of X remain continuous.
2. In more generality, if X is a scheme of finite type over $\mathbb{C}$, let $\mathcal{U} = \{U_i\}$ be an open cover, such that $U_i \cong \mathrm{Spec}(A_i)$ where A_i is an algebra of finite type of $\mathbb{C}$. Then $A_i = \mathbb{C}[X_1, ..., X_n]/(f_1, ..., f_r)$

where the f_j are polynomials in the variables $X_1, ..., X_n$ and hence holomorphic functions. So their zero locus is an analytic set U_i^{an}. Glueing these analytic sets, we obtain the complex analytic space X^{an} associated to X (this is not necessarily Hausdorff, see 3. below). We see now that, instead of the identity map in 1., we have a continuous map $\phi : X^{an} \longrightarrow X$ which maps X^{an} bijectively to the set of closed points of X.
3. We state some basic properties of the relation between X and X^{an}. We do this for X a scheme of finite type over $\mathbb{C}$. We have: X separated $\Leftrightarrow$ X^{an} Hausdorff, X connected $\Leftrightarrow$ X^{an} connected, X reduced $\Leftrightarrow$ X^{an} reduced, X smooth $\Leftrightarrow$ X^{an} complex manifold, X proper over $\mathbb{C}$ $\Leftrightarrow$ X^{an} compact.

Construction of $\mathcal{F}^{an}$ Consider the continuous map $id_X : X^{an} \longrightarrow X$. Let $\mathcal{F}' = {id_X}^{-1}(\mathcal{F})$ denote the inverse image sheaf of $\mathcal{F}$ under this map. Similarly, we have the sheaf ${\mathcal{O}_X}'$ on X^{an}. Then the sheaf $\mathcal{F}^{an} = \mathcal{F}' \otimes_{{\mathcal{O}_X}'} \mathcal{H}_{X^{an}}$ is called the **analytic sheaf** on X^{an} associated to $\mathcal{F}$.

Remark We see that $\mathcal{F}^{an}$ is a sheaf of $\mathcal{H}_{X^{an}}$-modules, so indeed an analytic sheaf, according to Definition 3.2.5. Also, as we would naturally expect, ${\mathcal{O}_X}^{an} = \mathcal{H}_{X^{an}}$.

The inclusion ${\mathcal{O}_X}' \longrightarrow \mathcal{H}_{X^{an}}$ gives rise to a canonical sheaf morphism $\alpha : \mathcal{F}' \longrightarrow \mathcal{F}^{an}$.
We notice now that every algebraic morphism of $\mathcal{O}_X$-modules $\phi : \mathcal{F} \longrightarrow \mathcal{G}$ gives rise to a morphism of ${\mathcal{O}_X}'$-modules $\phi' : \mathcal{F}' \longrightarrow \mathcal{G}'$ and then, upon tensoring with $\mathcal{H}_{X^{an}}$, to a morphism $\phi^{an} : \mathcal{F}^{an} \longrightarrow \mathcal{G}^{an}$. Hence analytification $\mathcal{F} \mapsto \mathcal{F}^{an}$ is a covariant functor. We now prove some of its properties.

Proposition 3.2.7. 1. The functor $\mathcal{F} \mapsto \mathcal{F}^{an}$ is exact.
2. For every algebraic sheaf $\mathcal{F}$, the morphism $\alpha : \mathcal{F}' \longrightarrow \mathcal{F}^{an}$ is injective.
3. If $\mathcal{F}$ is a coherent algebraic sheaf, $\mathcal{F}^{an}$ is a coherent analytic sheaf.

Proof Set $\mathcal{O} = \mathcal{O}_X$ and $\mathcal{H} = \mathcal{H}_{X^{an}} = \mathcal{O}^{an}$ for brevity. Suppose $\mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3$ is an exact sequence of algebraic sheaves on X, then the same is true for the sequence ${\mathcal{F}_1}' \longrightarrow {\mathcal{F}_2}' \longrightarrow {\mathcal{F}_3}'$. Now, we know that the pair of rings $(\mathcal{H}_x, \mathcal{O}_x)$ is a flat couple and hence in particular $\mathcal{H}_x$ is $\mathcal{O}_x$-flat (for a proof and more details, look at [1, no 6]). Therefore, tensoring with $\mathcal{H}$ over $\mathcal{O}$ (note that the stalks $\mathcal{O}'_x$ and $\mathcal{O}_x$ are equal) preserves exactness whence the sequence

$$ {\mathcal{F}_1}' \otimes_{\mathcal{O}'} \mathcal{H} \longrightarrow {\mathcal{F}_2}' \otimes_{\mathcal{O}'} \mathcal{H} \longrightarrow {\mathcal{F}_3}' \otimes_{\mathcal{O}'} \mathcal{H} $$

is exact, which finishes the proof of 1.
The proof of 2 follows in a similar fashion from the same fact about flatness of the couple $(\mathcal{H}_x, \mathcal{O}_x)$.
We proceed to the proof of 3. Let $\mathcal{O}^m$ denote the direct sum of m copies of $\mathcal{O}$. Since $\mathcal{F}$ is coherent, we have an exact sequence $\mathcal{O}^p \longrightarrow \mathcal{O}^q \longrightarrow \mathcal{F} \longrightarrow 0$, where the sheaves are restricted to a Zariski open set U. By 1, we get an exact sequence (recall $\mathcal{O}^{an} = \mathcal{H}$)

$$ \mathcal{H}^p \longrightarrow \mathcal{H}^q \longrightarrow \mathcal{F}^{an} \longrightarrow 0 $$

valid on U, which is open in the analytic topology (recall the analytic topology is finer than the Zariski topology). Since by Lemma 3.2.6 $\mathcal{H}$ is coherent, we conclude that $\mathcal{F}^{an}$ is coherent, as desired (cf. Section 2 or [2]). □

We finish this introductory session by stating another property of the analytification functor: its commutativity with the extension by zero functor. Let Y be a Zariski closed subvariety of an algebraic variety X and $\mathcal{F}$ a coherent algebraic sheaf on Y. We can extend $\mathcal{F}$ by zero on $X \setminus Y$ to obtain a coherent algebraic sheaf $\mathcal{F}^X$ on X and then a coherent analytic sheaf $(\mathcal{F}^X)^{an}$ on X^{an}. Since $\mathcal{F}^{an}$ is a coherent analytic sheaf on Y^{an} which is a closed subspace of X^{an} we can run the same procedure to obtain a coherent analytic sheaf $(\mathcal{F}^{an})^{X^{an}}$ on X^{an}. We then have

Proposition 3.2.8. [1, Proposition 11] The sheaves $(\mathcal{F}^X)^{an}$ and $(\mathcal{F}^{an})^X$ are canonically isomorphic.

Having said all the above preliminaries, this section is completed and we are now in a position to state GAGA and discuss its proof and applications.

3.3 Statement of GAGA

We state the GAGA principle/correspondence, as first written by Serre in his paper.

Theorem 3.3. (GAGA) Let X be a projective variety over $\mathbb{C}$ and X^{an} the associated complex analytic space. If $\mathcal{F}$ is a coherent algebraic sheaf on X, let $\mathcal{F}^{an}$ denote the corresponding (coherent) analytic sheaf. Then the following are true:

- 1. For every coherent algebraic sheaf $\mathcal{F}$ on X and every integer $q \geq 0$ we have a canonical isomorphism on cohomology $H^q(X, \mathcal{F}) \cong H^q(X^{an}, \mathcal{F}^{an})$.
- 2. If $\mathcal{F}, \mathcal{G}$ are two coherent algebraic sheaves on X, every analytic sheaf morphism from $\mathcal{F}^{an}$ to $\mathcal{G}^{an}$ comes from a unique algebraic sheaf morphism from $\mathcal{F}$ to $\mathcal{G}$.
- 3. Every coherent analytic sheaf $\mathcal{E}$ on X^{an} is isomorphic to $\mathcal{F}^{an}$ for some coherent algebraic sheaf $\mathcal{F}$ on X uniquely determined up to isomorphism.

Remark We note that Statements 2 & 3 together give an equivalence between the categories of coherent algebraic sheaves on X and coherent analytic sheaves on X^{an}.

In the next sections of this essay, we will mostly make use of the first Statement in the GAGA theorem, namely the invariance of cohomology. We choose thus to present a complete proof of 1, while giving a brief outline of the proofs of 2 and 3 (this holds true especially for the proof of 3, where we omit a lengthy proof of a theorem of Cartan).

3.4 Proof of GAGA (3.3.1.)

We will construct a natural homomorphism $\epsilon : \ H^q(X, \mathcal{F}) \longrightarrow H^q(X^{an}, \mathcal{F}^{an})$.
We claim that it is an isomorphism when X is projective and $\mathcal{F}$ is coherent (†).

Step 1: Construction of the map ϵ
We do this in full generality, i.e. for X an algebraic variety and $\mathcal{F}$ any algebraic sheaf on X, following Serre.
Recall that the identity map $id_X : X^{an} \longrightarrow X$ is continuous (by definition of X^{an}). For any Zariski open subset $U \subseteq X$ and s a section of $\mathcal{F}$ over U, we may view s as a section s' of the inverse image sheaf $\mathcal{F}' = {id_X}^{-1}(\mathcal{F})$ on X^{an} over U^{an}. Then we obtain a section $\epsilon(s) = s' \otimes 1$ of $\mathcal{F}^{an} = \mathcal{F}' \otimes \mathcal{H}$ over U^{an}. We thus have a map

$$\epsilon : \Gamma(U, \mathcal{F}) \longrightarrow \Gamma(U^{an}, \mathcal{F}^{an})$$

It is not difficult to see that this map induces a map on cohomology, using the setup of Čech cohomology. If $\mathcal{U} = \{U_i\}$ is a finite cover of X by Zariski open subsets, then $\mathcal{U}^{an} := \{{U_i}^{an}\}$ is a corresponding cover of X^{an} by open subsets. For all combinations of indices $i_0, i_1, ..., i_q$ we have then

$$\epsilon : \Gamma(U_{i_0} \cap ... \cap U_{i_q}, \mathcal{F}) \longrightarrow \Gamma(U_{i_0}^{an} \cap ... \cap U_{i_q}^{an}, \mathcal{F}^{an})$$

Hence we get homomorphisms between the corresponding groups of Čech $q-$cochains

$$\epsilon : C^q(\mathcal{U}, \mathcal{F}) \longrightarrow C^q(\mathcal{U}^{an}, \mathcal{F}^{an})$$

Now it is easy to notice that these maps commute with the Čech differential d and therefore they induce homomorphisms

$$\epsilon : H^q(\mathcal{U}, \mathcal{F}) \longrightarrow H^q(\mathcal{U}^{an}, \mathcal{F}^{an})$$

So taking the direct limit over open covers with respect to refinement or noticing that a cover by affine open subsets of X is a Leray cover, we finally obtain maps

$$\epsilon : H^q(X, \mathcal{F}) \longrightarrow H^q(X^{an}, \mathcal{F}^{an})$$

as desired.
We note that the naturality of this construction ensures that the constructed homomorphisms satisfy all functorial properties we would expect them to.

Step 2: Reduction to the case of projective space $\mathbb{P}^r(\mathbb{C})$
Since X is projective, there exists a closed embedding $i : X \longrightarrow \mathbb{P}^r(\mathbb{C})$ into complex projective space. We may identify $\mathcal{F}$ with the sheaf on $\mathbb{P}^r(\mathbb{C})$ obtained by extending it by zero outside X, thus viewing $\mathcal{F}$ as a coherent sheaf on $\mathbb{P}^r(\mathbb{C})$ supported on X. With this identification, we know that closed embeddings leave cohomology invariant (see [2, no 5, no 26] or [12]) and we therefore have for all $q \geq 0$: $H^q(X, \mathcal{F}) = H^q(\mathbb{P}^r(\mathbb{C}), \mathcal{F})$ and $H^q(X^{an}, \mathcal{F}^{an}) = H^q(\mathbb{P}^r(\mathbb{C})^{an}, \mathcal{F}^{an})$, where for the last equality to make perfect sense we have used the fact that analytification and extension by zero of a sheaf "commute" as sheaf operations and thus made the obvious identification up to canonical isomorphism (cf. Proposition 3.2.8.). Since the constructed maps in Step 1 are functorial, the following diagram commutes:

$$\begin{array}{ccc} H^q(X, \mathcal{F}) & \xrightarrow{\epsilon} & H^q(X^{an}, \mathcal{F}^{an}) \\ \cong \uparrow & & \uparrow \cong \\ H^q(\mathbb{P}^r(\mathbb{C}), \mathcal{F}) & \xrightarrow{\epsilon} & H^q(\mathbb{P}^r(\mathbb{C})^{an}, \mathcal{F}^{an}) \end{array}$$

From this, it is now clear that it is sufficient to prove our claim for the case $X = \mathbb{P}^r(\mathbb{C})$ only.

Step 3: Proof of (†) for the structure sheaf $\mathcal{O} = \mathcal{O}_{\mathbb{P}^r(\mathbb{C})}$
We set $X = \mathbb{P}^r(\mathbb{C})$.
For $q = 0$ it is a well-known fact that the only global sections of $\mathcal{O}$, $\mathcal{O}^{an}$ are constants and hence $H^0(X, \mathcal{O}) = H^0(X^{an}, \mathcal{O}^{an}) = \mathbb{C}$, as desired.
For $q > 0$, we know that $H^q(X, \mathcal{O}) = 0$. Now, by Dolbeault's Theorem, we have $H^q(X^{an}, \mathcal{O}^{an}) \cong H_{\bar{\partial}}{}^{0,q}(X^{an}) = 0$ (this is a known fact, proved also in Proposition 5.2.8.). So, we are done.

Step 4: Proof of (†) for the twisted structure sheaf $\mathcal{O}(n)$
Again setting $X = \mathbb{P}^r(\mathbb{C})$, we prove the claim by induction on r. For $r = 0$ there is nothing to prove.
Now let H be a non-zero linear form on the homogeneous coordinates $X_0, ..., X_r$, whose zero locus defines a hyperplane E. Then we have an exact sequence:

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_E \longrightarrow 0$$

where the map $\mathcal{O}(-1) \longrightarrow \mathcal{O}$ is multiplication by H and the map $\mathcal{O} \longrightarrow \mathcal{O}_E$ is just the restriction homomorphism. By tensoring with $\mathcal{O}(n)$ we obtain another exact sequence:

$$0 \longrightarrow \mathcal{O}(n-1) \longrightarrow \mathcal{O}(n) \longrightarrow \mathcal{O}_E(n) \longrightarrow 0$$

Taking the long exact sequence of cohomology corresponding to this short exact sequence and exploiting again the functorial properties of the maps defined in Step 1, we get the following commutative diagram:

$$\begin{array}{ccccccccc} \dots \to & H^q(X,\mathcal{O}(n-1)) & \to & H^q(X,\mathcal{O}(n)) & \to & H^q(X,\mathcal{O}_E(n)) & \to & H^{q+1}(X,\mathcal{O}(n-1)) & \to \dots \\ & \downarrow{\scriptstyle\epsilon} & & \downarrow{\scriptstyle\epsilon} & & \downarrow{\scriptstyle\epsilon} & & \downarrow{\scriptstyle\epsilon} & \\ \dots \to & H^q(X^{an},\mathcal{O}(n-1)^{an}) & \to & H^q(X^{an},\mathcal{O}(n)^{an}) & \to & H^q(X^{an},\mathcal{O}_E(n)^{an}) & \to & H^{q+1}(X^{an},\mathcal{O}(n-1)^{an}) & \to \dots \end{array}$$

Clearly E is isomorphic to $\mathbb{P}^{r-1}(\mathbb{C})$ and hence by the inductive hypothesis, the vertical arrows $\epsilon : H^q(X,\mathcal{O}(n)) \longrightarrow H^q(X^{an},\mathcal{O}(n)^{an})$ are isomorphisms for all $q \geq 0$ and integers n. Applying the Five Lemma, we conclude that our claim is true for $\mathcal{O}(n)$ if and only if it is true for $\mathcal{O}(n-1)$. Since by Step 3 the claim holds for $n = 0$, it holds for all n and we are done.

Step 5: Proof of (†) for every coherent algebraic sheaf $\mathcal{F}$
Set as usual $X = \mathbb{P}^r(\mathbb{C})$. We will establish the claim by descending induction on q. We can initiate the induction from any $q > 2r$ as then we know that both $H^q(X,\mathcal{F})$ and $H^q(X^{an},\mathcal{F}^{an})$ are equal to zero. It is a fact (recall Lemma 2.1.4.) that there exists an exact sequence of coherent algebraic sheaves

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{F} \longrightarrow 0$$

where $\mathcal{B}$ is a direct sum of sheaves isomorphic to $\mathcal{O}(n)$ for some integer n. So by Step 4, (†) holds for $\mathcal{B}$. Passing now to the long exact sequence of cohomology and again exploiting the functorial nature of the map ϵ we get a commutative diagram:

$$\begin{array}{ccccccccccc} \dots \to & H^q(X,\mathcal{A}) & \to & H^q(X,\mathcal{B}) & \to & H^q(X,\mathcal{F}) & \to & H^{q+1}(X,\mathcal{A}) & \to & H^{q+1}(X,\mathcal{B}) & \to \dots \\ & \downarrow{\scriptstyle\epsilon_1} & & \downarrow{\scriptstyle\epsilon_2} & & \downarrow{\scriptstyle\epsilon_3} & & \downarrow{\scriptstyle\epsilon_4} & & \downarrow{\scriptstyle\epsilon_5} & \\ \dots \to & H^q(X^{an},\mathcal{A}^{an}) & \to & H^q(X^{an},\mathcal{B}^{an}) & \to & H^q(X^{an},\mathcal{F}^{an}) & \to & H^{q+1}(X^{an},\mathcal{A}^{an}) & \to & H^{q+1}(X^{an},\mathcal{B}^{an}) & \to \dots \end{array}$$

Since (†) holds for $\mathcal{B}$ the maps ϵ_2, ϵ_5 are isomorphisms. By induction, so is the map ϵ_4. An application of one of the two Four Lemmas yields thus that ϵ_3 is surjective. Since $\mathcal{F}$ was arbitrary, the same applies to the coherent sheaf $\mathcal{A}$ and therefore ϵ_1 is also surjective. Now an application of the other Four Lemma shows that ϵ_3 is injective. Combining these two we obtain that ϵ_3 is an isomorphism, as desired. So Step 5 is completed and so is the proof of (†). □

Remarks At this point we just make some comments on the proof. We see that Step 1 is fairly natural and quite clearly motivated by the definitions of all the related objects. Step 2 makes use of a standard fact about cohomology to facilitate the rest of the discussion. The fact that every coherent algebraic sheaf is isomorphic to a quotient of a direct sum of twisted structure sheaves leads us to proceed to Steps 3 and 4 and is central to the proof of the last step, which concludes the proof.

3.5 Proof of GAGA (3.3.2.)

Set $\mathcal{O} = \mathcal{O}_X$ and $\mathcal{H} = \mathcal{H}_{X^{an}}$ for brevity.
Let $\mathcal{A} = \mathrm{Hom}(\mathcal{F},\mathcal{G})$ be the homomorphism sheaf from $\mathcal{F}$ to $\mathcal{G}$. Then by applying the analytification functor, we may obtain a morphism $\mathcal{A} \longrightarrow \mathrm{Hom}(\mathcal{F}^{\mathrm{an}},\mathcal{G}^{\mathrm{an}}) =: \mathcal{B}$. More precisely, if $f \in \mathcal{A}_x$ is a germ of a homomorphism from $\mathcal{F}$ to $\mathcal{G}$, we get a germ of a homomorphism f^{an} from $\mathcal{F}^{an}$ to $\mathcal{G}^{an}$. This map $f \mapsto f^{an}$ gives an $\mathcal{O}'$-linear morphism of the sheaf $\mathcal{A}'$ into the sheaf $\mathcal{B}$. Finally, applying the functor $\otimes_{\mathcal{O}'}\mathcal{H}$ we obtain a morphism $\iota : \mathcal{A}^{an} \longrightarrow \mathcal{B}$.

Claim $\iota : \mathcal{A}^{an} \longrightarrow \mathcal{B}$ is an isomorphism.

Proof For $x \in X$ we have by Proposition 3.1.3./3 of the Introduction that $\mathcal{A}_x = \mathrm{Hom}(\mathcal{F}_\mathrm{x}, \mathcal{G}_\mathrm{x})$ and hence $\mathcal{A}_x^{an} = \mathrm{Hom}(\mathcal{F}_\mathrm{x}, \mathcal{G}_\mathrm{x}) \otimes \mathcal{H}_\mathrm{x}$, where the functors $\otimes$ and Hom are applied over the ring $\mathcal{O}_x = \mathcal{O}'_x$.
Similarly, $\mathcal{F}^{an}$ is coherent and therefore $\mathcal{B}_x = \mathrm{Hom}(\mathcal{F}_\mathrm{x} \otimes \mathcal{H}_\mathrm{x}, \mathcal{G}_\mathrm{x} \otimes \mathcal{H}_\mathrm{x})$, the functor $\otimes$ applied over $\mathcal{O}_x$ and the functor Hom over $\mathcal{H}_x$ (recall the appropriate equalities of stalks of algebraic and the corresponding analytic sheafs).
Now we observe that the morphism

$$\iota_x : \mathrm{Hom}(\mathcal{F}_\mathrm{x}, \mathcal{G}_\mathrm{x}) \otimes \mathcal{H}_\mathrm{x} \longrightarrow \mathrm{Hom}(\mathcal{F}_\mathrm{x} \otimes \mathcal{H}_\mathrm{x}, \mathcal{G}_\mathrm{x} \otimes \mathcal{H}_\mathrm{x})$$

is in fact an isomorphism. This is a consequence of the fact that the pair $(\mathcal{H}_x, \mathcal{O}_x)$ is a flat couple and [1, Proposition 21]. For a more detailed account, see [1, Proposition 21]. So the proof of the claim is complete.

Having shown this, the rest of the proof is easy. Consider the homomorphisms

$$H^0(X, \mathcal{A}) \xrightarrow{\epsilon} H^0(X^{an}, \mathcal{A}^{an}) \xrightarrow{\iota} H^0(X^{an}, \mathcal{B})$$

Since the global sections of $\mathcal{A}$ (respectively $\mathcal{B}$) are algebraic morphisms from $\mathcal{F}$ to $\mathcal{G}$ (respectively analytic morphisms from $\mathcal{A}^{an}$ to $\mathcal{B}^{an}$) and for $f \in H^0(X, \mathcal{A})$ we have $\iota \circ \epsilon(f) = f^{an}$, (3.3.2.) is reduced to showing that the composition $\iota \circ \epsilon$ is an isomorphism. But this now is evident, as ϵ is an isomorphism by (3.3.1.) ($\mathcal{A}$ is coherent by Proposition 2.1.3./4 of the Introduction) and so is ι by our claim above. So we are done and (3.3.2.) is proved.

3.6 Sketch proof of GAGA (3.3.3.)

We first establish the uniqueness of $\mathcal{F}$. We see that it is a simple consequence of Statement 2 of the GAGA Theorem.
Suppose that $\mathcal{F}$ and $\mathcal{G}$ are two coherent algebraic sheaves satisfying $\mathcal{F}^{an} \cong \mathcal{E}$ and $\mathcal{G}^{an} \cong \mathcal{E}$. Then we have an isomorphism $\phi : \mathcal{F}^{an} \longrightarrow \mathcal{G}^{an}$. By (3.3.2) we get that there exists an algebraic sheaf morphism $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ such that $\phi = \psi^{an}$. Let $\mathcal{K}$ and $\mathcal{C}$ be the kernel and cokernel of ψ respectively. Then we get an exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{F} \xrightarrow{\psi} \mathcal{G} \longrightarrow \mathcal{C} \longrightarrow 0$$

Since analytification is exact, we get another exact sequence

$$0 \longrightarrow \mathcal{K}^{an} \longrightarrow \mathcal{F}^{an} \xrightarrow{\phi = \psi^{an}} \mathcal{G}^{an} \longrightarrow \mathcal{C}^{an} \longrightarrow 0$$

Since ϕ is an isomorphism we obtain that $\mathcal{K}^{an} = \mathcal{C}^{an} = 0$. But this implies that $\mathcal{K} = \mathcal{C} = 0$ by an application of Proposition 3.2.7./2. Therefore, we get that ϕ is an isomorphism, which proves uniqueness.

We press on to establish existence.

Step 1: Reduction to the case of projective space $\mathbb{P}^r(\mathbb{C})$
As in Step 2 in the proof of (3.3.1.), let $i : X \longrightarrow \mathbb{P}^r(\mathbb{C})$ be a closed embedding of X into complex projective space. Set $Y = \mathbb{P}^r(\mathbb{C})$ for brevity. Let also $\mathcal{E}$ be a coherent analytic sheaf on X^{an}. If the Statement we seek to prove is true for projective space, then there exists a coherent algebraic sheaf $\mathcal{G}$ on $\mathbb{P}^r(\mathbb{C})$ such that the coherent sheaf $\mathcal{E}^Y$ is isomorphic to $\mathcal{G}^{an}$. Let $\mathcal{I} = \mathcal{I}(X)$ be the coherent sheaf of ideals defined by the variety X. Let $f \in \mathcal{I}_x$ and ϕ the endomorphism of $\mathcal{G}_x$ given by multiplication by f. Since $\mathcal{E}$ is supported on X, it is clear that

the endomorphism ϕ^{an} of ${\mathcal{G}_x}^{an} = {\mathcal{E}_x}^Y = \mathcal{E}_x$ is equal to zero. The same is thus true (cf. Proposition 3.2.7./2) for ϕ and therefore $\mathcal{I} \cdot \mathcal{G} = 0$. This in turn implies that $\mathcal{G}$ is the extension by zero of some coherent algebraic sheaf $\mathcal{F}$ on X, i.e. $\mathcal{G} = \mathcal{F}^Y$. Hence, by Proposition 3.2.8. we obtain $\mathcal{E}^Y \cong \mathcal{G}^{an} = (\mathcal{F}^Y)^{an} \cong (\mathcal{F}^{an})^Y$, which by restriction to X gives the desired isomorphism $\mathcal{E} \cong \mathcal{F}^{an}$. So we are done and it suffices to prove the Statement when $X = \mathbb{P}^r(\mathbb{C})$.

From now on we assume that X stands for the complex projective space $\mathbb{P}^r(\mathbb{C})$. The proof will be by induction on r.

Step 2: Cartan's Theorems A and B
If $\mathcal{M}$ is any analytic sheaf on X, we may define for any integer n the twisted analytic sheaf $\mathcal{M}(n)$ in exactly the same way as the twisted algebraic sheaf $\mathcal{O}(n)$: Let $\mathcal{U} = \{U_i\}$ be the standard open cover of X by $r+1$ affine sets. Then multiplication by ${X_j}^n/{X_i}^n$ is an isomorphism of the sheaves $\mathcal{M}|_{U_i}$ and $\mathcal{M}|_{U_j}$ restricted to $U_i \cap U_j$. These isomorphisms satisfy the appropriate cocycle condition, so we can reglue them to obtain the sheaf $\mathcal{M}(n)$. Clearly $\mathcal{M}(n)$ is coherent when $\mathcal{M}$ is, being locally isomorphic to $\mathcal{M}$. Also, we have as in the algebraic case $\mathcal{M}(n) = \mathcal{M} \otimes_{\mathcal{H}} \mathcal{H}(n)$, where $\mathcal{H} = \mathcal{H}_X$. Finally for an algebraic sheaf $\mathcal{F}$, $\mathcal{F}^{an}(n) = \mathcal{F}(n)^{an}$.

We now state an appropriate version of Cartan's Theorems A and B, omitting their proofs (we refer the interested reader to [1, no 16]). The statement is the same to be found there. A more general version for compact complex manifolds can be found in [8, p. 700].

Theorem A Let L a hyperplane of $\mathbb{P}^r(\mathbb{C})$ and $\mathcal{A}$ a coherent analytic sheaf on L. Then $H^q(L^{an}, \mathcal{A}(n)) = 0$ for all $q > 0$ and n sufficiently large.

Theorem B Let $\mathcal{M}$ be a coherent analytic sheaf on $X = \mathbb{P}^r(\mathbb{C})$. Then there exists $n(\mathcal{M}) \in \mathbb{Z}$ such that, for all $n \geq n(\mathcal{M})$ and all $x \in X$, the $\mathcal{H}_x$-module $\mathcal{M}(n)_x$ is generated by elements of $H^0(X^{an}, \mathcal{M}(n))$, i.e. $\mathcal{M}(n)$ is spanned by its global sections.

Given the inductive hypothesis (notice $L \cong \mathbb{P}^{r-1}(\mathbb{C})$) and Statement 3.3.1., Theorem A is actually fairly easy to show as it reduces to the analogous vanishing theorem for coherent algebraic sheaves on projective space by Serre (cf. Theorem 2.3.1./2). Theorem A is used in the proof of Theorem B in Serre's paper and this is the only reason it appears in this exposition. Theorem B is the core of the proof of 3.3.3. We will see that the rest of the proof bears no significant difficulties.

Step 3: Proof of (3.3.3.) for projective space
Let $\mathcal{E}$ be a coherent analytic sheaf on $X = \mathbb{P}^r(\mathbb{C})$. As usual, $\mathcal{H} = \mathcal{H}_X$. By Theorem B, we get that for some integer n, $\mathcal{E}(n)$ is isomorphic to a quotient sheaf of the sheaf $\mathcal{H}^p$ (direct sum of p copies of $\mathcal{H}$), taking into account also the fact that $H^0(X^{an}, \mathcal{M}(n))$ has finite dimension as a complex vector space. Hence by the definition of the twisted sheaf it follows that $\mathcal{E}$ is isomorphic to a quotient sheaf of $\mathcal{H}(-n)^p$. Setting $\mathcal{L}_0 = \mathcal{O}(-n)^p$ we obtain an exact sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{L}_0^{an} \longrightarrow \mathcal{E} \longrightarrow 0$$

where $\mathcal{A}$ is a coherent analytic sheaf. Repeating this procedure for $\mathcal{A}$ we may find a coherent algebraic sheaf $\mathcal{L}_1$ and a surjection $\mathcal{L}_1^{an} \longrightarrow \mathcal{A}$. So we get an exact sequence

$$\mathcal{L}_1^{an} \xrightarrow{g} \mathcal{L}_0^{an} \longrightarrow \mathcal{E} \longrightarrow 0$$

Now by (3.3.2.) there exists an algebraic morphism $f : \mathcal{L}_1 \longrightarrow \mathcal{L}_0$ such that $g = f^{an}$. So, if $\mathcal{F}$ is the cokernel of f, we have an exact sequence

$$\mathcal{L}_1 \xrightarrow{f} \mathcal{L}_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

and by applying the exact analytification functor an exact sequence

$$\mathcal{L}_1^{an} \xrightarrow{g} \mathcal{L}_0^{an} \longrightarrow \mathcal{F}^{an} \longrightarrow 0$$

Comparing with the above we see that $\mathcal{E}$ is isomorphic to $\mathcal{F}^{an}$ and since $\mathcal{F}$ is a coherent algebraic sheaf, the proof is complete.

Remark We have named this a sketch proof, even though we have just omitted the proof of Theorem B in the form stated. However, the proof of Theorem B is quite long and probably constitutes the hardest part of the whole proof, which otherwise is not particularly hard.

3.7 Applications of GAGA and some theorems

We now proceed to exhibit a few applications of GAGA as a brief illustration of its power and elegance. We will also state and prove some results which fit well in the interplay between algebraic and analytic properties, but do not necessarily follow from GAGA or use it.

We begin by stating without proof some neat results relating properties of the Zariski and analytic topology. These can be found in Serre's paper [1] (we refer the reader there for their proofs as well). Except for their elegance, they appear because they will be useful in applications of Chow's Theorem.

Proposition 3.7.1. If $f : X \longrightarrow Y$ is a regular map between algebraic varieties, then the analytic closure and Zariski closure of $f(X)$ are equal.

Proposition 3.7.2. Let X, Y be two algebraic varieties and $f : X \longrightarrow Y$ a holomorphic map from X to Y. If the graph of f is a subvariety of $X \times Y$, then $f : X \longrightarrow Y$ is a regular map.

We continue by proving the well-known theorem of Chow.

Theorem 3.7.3. (Chow) Every closed analytic subset of complex projective space is an algebraic variety.

Proof Set $Y = \mathbb{P}^r(\mathbb{C})$. Suppose that X is a closed analytic subset of Y^{an}. By a theorem of H. Cartan, the analytic sheaf $\mathcal{H}_X = \mathcal{H}_Y/\mathcal{A}(X)$ on Y_{an} is coherent and is supported on X. Hence by Statement 3 of GAGA, $\mathcal{H}_X = \mathcal{F}^{an}$ for some coherent algebraic sheaf $\mathcal{F}$ on Y. Now, the supports of $\mathcal{F}$ and $\mathcal{F}^{an}$ are clearly equal. Hence, since the support of $\mathcal{F}$ is Zariski closed in Y, as $\mathcal{F}$ is coherent, we deduce that X is Zariski closed in Y and hence algebraic. □

Remark We see that the proof of Chow's Theorem was quite easy, given all our earlier work and the strength of the GAGA Theorem.

An elegant corollary of Chow's Theorem is the following special case:

Corollary 3.7.4. Every projective complex manifold is algebraic.

The following are two applications of Chow's Theorem.

Proposition 3.7.5. If X is a complex algebraic variety, every compact analytic subset $X' \subseteq X$ is algebraic.

Proof We make use of the following fact (originally a result of Chow, cf. proof of [1, Proposition 6]):
There exists a projective variety Y, a Zariski open and dense subset $U \subseteq Y$ and a surjective regular map $f : U \longrightarrow X$ whose graph T is Zariski closed in $X \times Y$.
Set $T' = T \cap (X' \times Y)$. X', Y are compact (Y is projective) and T is closed (it is Zariski closed and the analytic topology is finer) so we obtain that T' is compact. Hence if $\mathrm{pr}_2 : \mathrm{T}' \longrightarrow \mathrm{Y}$ is the projection onto the second factor, $Y' = \mathrm{pr}_2(\mathrm{T}')$ is a compact subset of Y. Now notice that $Y' = f^{-1}(X')$ and hence Y' is an analytic subset of U and therefore of Y. Now Chow's Theorem shows that Y' is a closed subvariety of Y. We are now in position to apply Proposition 3.7.1. to the regular map $f : Y' \longrightarrow X$. From this we conclude that $X' = f(Y')$ is Zariski closed in X, i.e. algebraic, and we are done. □

Proposition 3.7.6. If $f : X \longrightarrow Y$ is a holomorphic map from a compact algebraic variety X to an algebraic variety Y, then f is regular.

Proof Let T be the graph of f in $X \times Y$. Then f being holomorphic implies that T is a compact analytic subset of $X \times Y$. Hence by the previous Proposition 3.7.5., we get that T is in fact algebraic. Now an application of Proposition 3.7.2. shows that f is regular and we are done. □

Remarks We have closely followed Serre's original exposition in the above. We have systematically omitted the exponent "*an*" and always used the adjective "Zariski" when we refer to the Zariski topology. Hopefully there is no confusion for the reader and the way we view our objects (algebraic or analytic) is clear from the context.

We will see that another application of the GAGA principle will appear in the next section about Grothendieck's algebraic de Rham Theorem.

4 Grothendieck's algebraic de Rham Theorem

4.1 Hypercohomology: Definition and basic facts

In order to state and discuss Grothendieck's algebraic de Rham Theorem, it will be of use to give the basic definitions and properties of hypercohomology. We will follow closely the exposition in [8]. We will thus present the more concrete Čech setup of hypercohomology, which very much resembles the setup of classical Čech cohomology. For a treatment using injective resolutions and derived functors, we refer the interested reader to [9]. Of course, as usual, the two theories can be shown to be equivalent. A fair bit of spectral sequences will be used as well. The reader is encouraged to look at Section 5.3, which contains a brief account of the theory.

Definition Hypercohomology is a natural generalization of sheaf cohomology to complexes of sheaves.
Let X be a topological space. Let $(\mathcal{F}^\bullet, d)$ be a (bounded below) complex of sheaves of abelian groups on X $(d \circ d = 0)$

$$\mathcal{F}^0 \stackrel{d}{\to} \dots \stackrel{d}{\to} \mathcal{F}^p \stackrel{d}{\to} \mathcal{F}^{p+1} \stackrel{d}{\to} \dots$$

To this complex of sheaves we may associate the cohomology sheaves $\mathcal{H}^q = \mathcal{H}^q(\mathcal{F}^\bullet)$ in a natural way (see [8, pp. 445-446] for details). Now let $\mathcal{U} = \{U_\alpha\}$ be an open cover of X and $C^p(\mathcal{U}, \mathcal{F}^q)$ the group of Čech p-cochains with values in $\mathcal{F}^q$. Then the two operators (δ is the Čech differential)

$$\delta : C^p(\mathcal{U}, \mathcal{F}^q) \longrightarrow C^{p+1}(\mathcal{U}, \mathcal{F}^q)$$

$$d : C^p(\mathcal{U}, \mathcal{F}^q) \longrightarrow C^p(\mathcal{U}, \mathcal{F}^{q+1})$$

satisfy the identities $\delta^2 = d^2 = 0$, $d\delta + \delta d = 0$. Thus we obtain a bigraded complex $C^{p,q} = C^p(\mathcal{U}, \mathcal{F}^q)$ with differentials δ, d. Let $(C^\bullet, D)$, where $D = \delta + d$, be the associated single complex. A refinement $\mathcal{U}'$ of $\mathcal{U}$ induces maps

$$C^p(\mathcal{U}, \mathcal{F}^q) \longrightarrow C^p(\mathcal{U}', \mathcal{F}^q)$$

$$H^*(C^\bullet(\mathcal{U})) \longrightarrow H^*(C^\bullet(\mathcal{U}'))$$

Then we may define the **hypercohomology groups** $\mathbb{H}^n(X, \mathcal{F}^\bullet)$ as the direct limit over refinements of open covers

$$\mathbb{H}^*(X, \mathcal{F}^\bullet) = \varinjlim H^*(C^\bullet(\mathcal{U}), D)$$

Fact 1 If a map $j : \mathcal{A}^\bullet \longrightarrow \mathcal{B}^\bullet$ between complexes of sheaves induces isomorphisms on the cohomology sheaves $j_* : \mathcal{H}^q(\mathcal{A}^\bullet) \longrightarrow \mathcal{H}^q(\mathcal{B}^\bullet)$ (we then say that j is a **quasi-isomorphism**), then it also induces an isomorphism $j_* : \mathbb{H}^*(X, \mathcal{A}^\bullet) \longrightarrow \mathbb{H}^*(X, \mathcal{B}^\bullet)$ on hypercohomology. That is, quasi-isomorphisms leave hypercohomology invariant.

Fact 2 We know that there are two spectral sequences $'E$, $''E$ associated to the double complex $C^{p,q} = C^p(\mathcal{U}, \mathcal{F}^q)$ (cf. Section 5.3. for more details and basic properties). It is easy to see that they behave well with respect to refinements of open covers and so by taking the direct limit we obtain two spectral sequences $'E$, $''E$ which converge to $\mathbb{H}^*(X, \mathcal{F}^\bullet)$. Their second terms are

$$'E_2^{p,q} = H^p(X, \mathcal{H}^q(\mathcal{F}^\bullet))$$

$$''E_2^{p,q} = H^q_d(H^p(X, \mathcal{F}^\bullet))$$

4.2 The analytic and algebraic de Rham Theorems

First we establish some standard notation.
Let X be a non-singular scheme of finite type over $\mathbb{C}$, $(\Omega^\bullet_X, d)$ the complex of sheaves of regular differential forms and $(\Omega^\bullet_{X^{an}}, d)$ the complex of sheaves of holomorphic differential forms. Also, as usual, let $\underline{\mathbb{C}}$ be the constant sheaf of stalk $\mathbb{C}$ on X^{an}.

Definition 4.2.1. (Algebraic de Rham Cohomology) The algebraic de Rham cohomology $H^*_{dR}(X)$ of X is defined by the hypercohomology groups $H^*_{dR}(X) = \mathbb{H}^*(X, \Omega^\bullet_X)$.

We also have the classical **analytic de Rham cohomology** of X: $H^*_{dR}(X^{an}) = \mathbb{H}^*(X^{an}, \Omega^\bullet_{X^{an}})$.

We now give a short demonstration of the analytic de Rham Theorem and its proof.
The holomorphic Poincaré lemma shows that $\underline{\mathbb{C}} \longrightarrow \Omega^\bullet_{X^{an}}$ is a resolution of the constant sheaf $\underline{\mathbb{C}}$. This implies that if we let $\underline{\mathbb{C}}^\bullet$ be the complex $\underline{\mathbb{C}} \longrightarrow 0 \longrightarrow 0 \longrightarrow ...$ with $\underline{\mathbb{C}}$ in degree zero, we obtain a quasi-isomorphism $\underline{\mathbb{C}}^\bullet \longrightarrow \Omega^\bullet_{X^{an}}$ and therefore by Fact 1 of the previous section

$$\mathbb{H}^*(X^{an}, \underline{\mathbb{C}}^\bullet) \cong \mathbb{H}^*(X^{an}, \Omega^\bullet_{X^{an}}) = H^*_{dR}(X^{an})$$

Noticing now that clearly $H^*(X^{an}, \underline{\mathbb{C}})$ equals $\mathbb{H}^*(X^{an}, \underline{\mathbb{C}}^\bullet)$ and since this is further equal to the singular complex cohomology of X^{an}, $H^*(X^{an}, \mathbb{C})$ we finally obtain the

Theorem 4.2.2. (Analytic de Rham Theorem) $H^*(X^{an}, \mathbb{C}) = H^*(X^{an}, \underline{\mathbb{C}}) \cong H^*_{dR}(X^{an})$.

We have a canonical homomorphism of the algebraic into the analytic de Rham cohomology $H^*_{dR}(X) \longrightarrow H^*_{dR}(X^{an})$ which comes from viewing algebraic forms as holomorphic forms. We are now in a position to state a form of Grothendieck's algebraic de Rham Theorem (as appeared in his original paper [3]).

Theorem 4.2.3. (Grothendieck's algebraic de Rham Theorem) Let X be a non-singular scheme of finite type over $\mathbb{C}$. Then the following are true:

1. The canonical homomorphism $H^*_{dR}(X) \longrightarrow H^*_{dR}(X^{an}) \cong H^*(X^{an}, \underline{\mathbb{C}})$ is an isomorphism.
2. If X is affine, $H^*(\Gamma(X, \Omega^\bullet_X), d) \cong H^*_{dR}(X^{an}) \cong H^*(X^{an}, \underline{\mathbb{C}})$.

The moral behind Grothendieck's algebraic de Rham Theorem as well as its value lie in the observation that the right hand side in both Statements 1 & 2 is an inherent topological quantity, being the singular complex cohomology of X^{an}, whereas the left hand side comprises purely algebraic information. Often, the left hand side of Statement 2 can be explicitly calculated. So, we can compute the singular cohomology of a smooth variety by computing the hypercohomology of its algebraic de Rham complex. This is yet another example of how formal calculus can be useful in algebraic situations.

Examples 4.2.4. 1. (Punctured line) Let $X = \mathrm{Spec}(R)$ where $R = \mathbb{C}[z, \frac{1}{(z-z_1)...(z-z_k)}]$. This is the affine line $\mathbb{A}^1_{\mathbb{C}}$ with k punctures at the points $z_1, ..., z_k$. Then by (4.2.3./2) the de Rham cohomology is equal to the cohomology of the complex of global differential forms. A differential form in Ω^1_X can be written as $\frac{f(z)}{g(z)}dz$ where g has zeros possibly only at the z_i. Then it is clear that the first cohomology group has a basis given by the forms $\frac{1}{z-z_i}dz$ (everything else is clearly exact) and hence dim $H^1(X) = k$.
2. (Elliptic curve) Let X be a complex elliptic curve. X has dimension 1 and hence $\Omega^p_X = 0$ for all $p \geq 2$. To compute the algebraic de Rham cohomology, we will make use of the spectral sequence $''E_2^{p,q}$. We firstly calculate the terms $H^p(X, \Omega^q_X)$ for $p, q \leq 1$. Since X is projective, we know that $H^0(X, \Omega^0_X) = \mathbb{C}$. Moreover $g(X) = 1$ and thus $H^0(X, \Omega^1_X) = \mathbb{C}\omega$ for a

differential form ω. By Serre duality, we obtain also $H^1(X,\Omega^0_X) = H^0(X,\Omega^1_X)^\vee = (\mathbb{C}\omega)^\vee$ and $H^1(X,\Omega^1_X) = H^0(X,\Omega^0_X)^\vee = \mathbb{C}^\vee$. It is clear then that the spectral sequence degenerates at $''E_1^{p,q}$ as all the E_1-differentials are zero. Hence $''E_\infty^{p,q} = ''E_1{}^{p,q} = H^q(X,\Omega^p_X)$ and since the spectral sequence converges to $H^*_{dR}(X)$, we obtain that dim $H^i_{dR}(X) = 1,2,1$ for $i = 0,1,2$, which agrees to the singular complex cohomology, as X^{an} is a complex torus.

4.3 Proof of Grothendieck's algebraic de Rham Theorem

Step 1: Proof that (4.2.3./2) implies (4.2.3./1)
Suppose that X is affine. Recall Fact 2 from the introductory section 4.1. The second hypercohomology spectral sequence for $H^*_{dR}(X)$ reads

$$''E_1^{p,q} = H^q(X,\Omega^p_X) \Longrightarrow H^*_{dR}(X)$$

Since the sheaves Ω^p_X are coherent, Serre's vanishing principle (Theorem 2.3.2.) shows that $H^q(X,\Omega^p_X) = 0$ for all $q > 0$. This implies that the spectral sequence degenerates at its second term $''E_2$ and we evidently get that $H^*_{dR}(X)$ is the cohomology of the global section complex of the complex of sheaves $(\Omega^\bullet_X, d)$ ([9, Proposition 4.32]), i.e.

$$H^*_{dR}(X) = H^*(\Gamma(X,\Omega^\bullet_X), d)$$

This establishes the claim when X is affine (in fact we see that the two Statements of the theorem are then equivalent).

Now suppose that X is as in the statement of (4.2.3). Take an open affine cover $\mathcal{U} = \{U_i\}$ of X and let $\mathcal{U}^{an}$ be the corresponding cover of X^{an}. (We note that since by Serre's vanishing principle all coherent sheaves on the U_i are acyclic, the cover is a Leray cover and thus computes Čech (sheaf) cohomology. By GAGA, the same holds true for the cover $\mathcal{U}^{an}$ of X^{an}.) As before, write now $\mathcal{H}^q$ for the cohomology sheaf associated to the presheaf $V \longmapsto H^q_{dR}(V)$ on X and similarly $\mathcal{H}^{an,q}$ for $V \longmapsto H^q_{dR}(V)$ on X^{an}. By Fact 2 from Section 4.1, there are convergent spectral sequences on X and X^{an}

$$'E_2^{p,q} = H^p(\mathcal{U},\mathcal{H}^q) \Longrightarrow H^*_{dR}(X)$$

$$'E_2^{an,p,q} = H^p(\mathcal{U}^{an},\mathcal{H}^{an,q}) \Longrightarrow H^*_{dR}(X^{an})$$

These are related via the canonical maps

$$H^p(\mathcal{U},\mathcal{H}^q) \longrightarrow H^p(\mathcal{U}^{an},\mathcal{H}^{an,q}) \ (1)$$

$$H^*_{dR}(X) \longrightarrow H^*_{dR}(X^{an}) \ (2)$$

Hence, to show that map (2) is an isomorphism it suffices to show that the second terms of the two Čech-to-sheaf spectral sequences are isomorphic via the canonical map (1).
But, these terms involve cohomology groups over sets of the form $U_{i_0} \cap ... \cap U_{i_q}$ and we may thus assume that X is contained in an affine scheme and is therefore separated. X being separated implies that the intersections $U_{i_0} \cap ... \cap U_{i_q}$ are affine. We treated precisely this case earlier and so indeed (1) is an isomorphism and so is (2). We are done and the claim is shown. Thus Statement (4.2.3./1) follows in general from (4.2.3./2).

Comment If X is projective, then we can prove (4.2.3./1) straight away without assuming (4.2.3./2). This can be done by using the spectral sequence $''E_1^{p,q}$ for the algebraic and holomorphic de Rham complexes and applying GAGA to obtain the result immediately.

Step 2: Preparatory results
Before delving into the proof we discuss the ingredients that will be of use.
We state the following result without proof.

Theorem 4.3.1. (Corollary of Hironaka's Theorem on resolution of singularities) There exists a resolution of singularities $f : (Y', D', X') \longrightarrow (Y, D, X)$. This means that there exists a proper morphism of projective varieties $Y' \longrightarrow Y$ such that $f|_{X'} : X' \longrightarrow X$ is an isomorphism, Y' is non-singular and D' is a normal crossing divisor.

Now, let Y^{an} be a complex manifold and D a normal crossing divisor on Y^{an}. The fact that D is a normal crossing divisor is necessary for the proof of Theorem 4.3.2., which we omit. We thus do not give the definition of a normal crossing divisor either (intuitively, it is an intersection of hyperplanes locally). The interested reader can look at [9, Definition 8.15] or [8, p.449]. Set $U = Y^{an} - D$.
Let $\Omega^p_{Y^{an}}(nD)$ be the sheaf of meromorphic p-forms which are holomorphic on U and have polar singularities of order at most n along D. Then set

$$\Omega^p_{Y^{an}}(\infty D) = \varinjlim \Omega^p_{Y^{an}}(nD) = \bigcup_{n \geq 0} \Omega^p_{Y^{an}}(nD)$$

and $\Omega^\bullet_{Y^{an}}(\infty D)$ to be the associated complex of sheaves with the natural exterior differential.
Let $j : U \hookrightarrow Y^{an}$ be the inclusion map. If, as usual, $\mathcal{A}^p_U$ is the sheaf of smooth p-forms on U and $\mathcal{A}^\bullet_U$ the associated complex, then set $\mathcal{A}^\bullet_{Y^{an}}(\infty D) = j_* \mathcal{A}^\bullet_U$, the complex of sheaves of C^∞ forms on Y^{an} allowing arbitrary singularities along D.
We then have a natural inclusion

$$\Omega^\bullet_{Y^{an}}(\infty D) \subset \mathcal{A}^\bullet_{Y^{an}}(\infty D)$$

of complexes, which is compatible with the exterior differential, hence a morphism of complexes of sheaves.

Theorem 4.3.2. ([8, Lemma, p. 450]) The inclusion $\Omega^\bullet_{Y^{an}}(\infty D) \hookrightarrow \mathcal{A}^\bullet_{Y^{an}}(\infty D)$ is a quasi-isomorphism.

We will also make use of the following Lemma.

Lemma 4.3.3. There is a canonical isomorphism

$$H^*(U, \underline{\mathbb{C}}) = \mathbb{H}^*(Y^{an}, \Omega^\bullet_{Y^{an}}(\infty D))$$

Proof By Theorem 4.3.2. and Fact 1 of 4.1., we get that

$$\mathbb{H}^*(Y^{an}, \Omega^\bullet_{Y^{an}}(\infty D)) \cong \mathbb{H}^*(Y^{an}, \mathcal{A}^\bullet_{Y^{an}}(\infty D)) \quad (1)$$

We know that the sheaves $\mathcal{A}^p_{Y^{an}}(\infty D))$ are fine (by a standard partition of unity argument) and therefore acyclic (their higher cohomology vanishes, i.e. $H^q(Y^{an}, \mathcal{A}^p_{Y^{an}}(\infty D)) = 0$ for all $q > 0$). From this, it follows that

$$\mathbb{H}^*(Y^{an}, \mathcal{A}^\bullet_{Y^{an}}(\infty D)) = H^*(\Gamma(Y^{an}, \mathcal{A}^\bullet_{Y^{an}}(\infty D)), d) \quad (2)$$

By the definition of $\mathcal{A}^\bullet_{Y^{an}}(\infty D)$ it follows that $\Gamma(Y^{an}, \mathcal{A}^\bullet_{Y^{an}}(\infty D)) = \Gamma(U, \mathcal{A}^\bullet_U)$. Now, the standard smooth de Rham's theorem yields

$$H^*(\Gamma(U, \mathcal{A}^\bullet_U), d) = H^*(U, \underline{\mathbb{C}}) \quad (3)$$

So, combining (1),(2) and (3), we obtain the desired isomorphism. □

Comment The role of the complex $\Omega^\bullet_{Y^{an}}(\infty D)$ could also be played equally well by the logarithmic complex $\Omega^\bullet_{Y^{an}}(\infty\log \text{D})$, which is the complex of differential forms that are holomorphic on X and have logarithmic singularities along D. The interested reader can see [8] or [9] for more details.

Step 3: Proof of Statement (4.2.3./2)
Choose a projective closure $\iota : X \hookrightarrow Y$. Let D be the divisor $D = Y - X$.
By Theorem 4.3.1., we may choose a resolution of singularities $f : (Y', D', X') \longrightarrow (Y, D, X)$. This enables us to work with the pair (Y', D') rather than the pair (Y, D). Equivalently we may assume that Y is a non-singular projective variety and D is a normal crossing divisor. Note that smoothness of Y implies that Y^{an} has the structure of a complex manifold.
Now by commutativity of cohomology with direct limits and by applying GAGA (it is easy to check that the sheaves $\Omega^p_{Y^{an}}(nD)$ are coherent, even locally free) we get

$$H^q(Y^{an}, \Omega^p_{Y^{an}}(\infty D)) = \varinjlim H^q(Y^{an}, \Omega^p_{Y^{an}}(nD)) = \varinjlim H^q(Y, \Omega^p_Y(nD))$$

The sheaves $\Omega^p_Y(nD)$ are defined in the natural analogous way and are exactly what one would expect.
Now, an application of Theorem 2.3.1./2 shows that for n sufficiently large $H^q(Y, \Omega^p_Y(nD)) = 0$ for all $q > 0$. This implies by the above that $H^q(Y^{an}, \Omega^p_{Y^{an}}(\infty D)) = 0$ for $q > 0$. An alternative way to see this would be to notice straight away without using GAGA that for n sufficiently large $H^q(Y^{an}, \Omega^p_{Y^{an}}(nD)) = 0$ for all $q > 0$ follows from applying Cartan's Theorem B from Section 3.6.
Therefore (recall Fact 2 from Section 4.1), the hypercohomology spectral sequence

$$''E_2^{p,q} = H^q_d(H^p(Y^{an}, \Omega^\bullet_{Y^{an}}(\infty D))) \Longrightarrow \mathbb{H}^*(Y^{an}, \Omega^\bullet_{Y^{an}}(\infty D))$$

degenerates at the second term and, as we have seen multiple times before, it follows that

$$\mathbb{H}^*(Y^{an}, \Omega^\bullet_{Y^{an}}(\infty D)) \cong H^*(\Gamma(Y^{an}, \Omega^\bullet_{Y^{an}}(\infty D)), d)$$

So, taking into account Lemma 4.3.3., we see that the left hand side of the above isomorphism equals $H^*(X^{an}, \underline{\mathbb{C}})$ and we are left with the task of showing that the right hand side is $H^*_{dR}(X)$.
Recall that in Step 1 we proved that $H^*_{dR}(X) = H^*(\Gamma(X, \Omega^\bullet_X), d)$ when X is affine. We have similarly as before

$$H^*(\Gamma(Y^{an}, \Omega^\bullet_{Y^{an}}(\infty D)), d) = \varinjlim H^*(\Gamma(Y^{an}, \Omega^\bullet_{Y^{an}}(nD)), d) = \varinjlim H^*(\Gamma(Y, \Omega^\bullet_Y(nD)), d)$$

So it will suffice to show that the restriction map

$$\varinjlim \Gamma(Y, \Omega^\bullet_Y(nD)) \longrightarrow \Gamma(X, \Omega^\bullet_X)$$

is an isomorphism.
Injectivity of this map is clear, while the surjectivity follows immediately from Serre's theorem on extensions of sections over projective varieties. So we are done and the proof is complete.

5 Hodge Theory

Our main reference for this section will be Voisin's book [9]. Many of our results will concern compact, Kähler manifolds. Hence they will also apply to smooth projective complex varieties, which are the objects we are primarily interested in.

5.1 The Hodge Theorem

In this introduction we will remind of the reader of some basic definitions and results, as well as some standard notation that will be used in the next sections. All this can be found in [8] or [9]. We prove a small selection of the exhibited results.

Let X be a compact complex manifold. We can equip X with a Hermitian metric ds^2. Let ω be the associated $(1,1)$ form. ds^2 induces a Hermitian inner product on all tensor bundles $T^{*(p,q)}(X)$ which we denote by $(\ ,\ )$. Let $\Phi = \omega^n/n!$ be the associated volume form.

Definition 5.1.1. (Hodge $*$-operator) Define the Hodge star operator $* : \mathcal{A}_X^{p,q} \longrightarrow \mathcal{A}_X^{n-p,n-q}$ by requiring that for all open subsets $U \subseteq X$ the following identity holds for all $\psi, \eta \in \mathcal{A}_X^{p,q}$

$$(\psi(z), \eta(z))\Phi(z) = \psi(z) \wedge *\eta(z)$$

For $\psi, \eta \in A^{p,q}(X)$ we can define a global L^2 inner product

$$(\psi, \eta) = \int_X (\psi(z), \eta(z))\Phi(z) = \int_X \psi(z) \wedge *\eta(z)$$

Also, we may define the **formal adjoint operator** $\bar{\vartheta}^*$ by $\bar{\vartheta}^* = - * \bar{\vartheta} *$. ϑ^* is defined in the same way. Then $d^* = (\vartheta + \bar{\vartheta})^* = \vartheta^* + \bar{\vartheta}^*$.

It is easy to show (simple algebraic manipulation) that $* * \psi = (-1)^{p+q}\psi$ and that $(\bar{\vartheta}, \bar{\vartheta}^*)$ and (ϑ, ϑ^*) are pairs of adjoint operators with respect to the L^2 product just defined. We have now the following standard definition.

Definition 5.1.2. (Laplacians) The $d, \vartheta, \bar{\vartheta}$-Laplacians $\Delta_d : A_X^r \longrightarrow A_X^r,\ \Delta_\vartheta, \Delta_{\bar{\vartheta}} : A_X^{p,q} \longrightarrow A_X^{p,q}$ are defined by

$$\Delta_d = dd^* + d^*d$$
$$\Delta_\vartheta = \vartheta\vartheta^* + \vartheta^*\vartheta$$
$$\Delta_{\bar{\vartheta}} = \bar{\vartheta}\bar{\vartheta}^* + \bar{\vartheta}^*\bar{\vartheta}$$

Let us denote $\Delta = \Delta_{\bar{\vartheta}}$, as this is the Laplacian we will mainly be concerned with in this section.

Definition 5.1.3. ($\bar{\vartheta}$-harmonic form) $\psi \in A_X^{p,q}$ is said to be **$\bar{\vartheta}$-harmonic** if $\Delta\psi = \Delta_{\bar{\vartheta}}\psi = 0$. Let $\mathcal{H}^{p,q} = \mathcal{H}_{\bar{\vartheta}}^{p,q} = \ker\Delta$ denote this space of harmonic forms.

Similar definitions apply to the other two operators (we denote the space of d-harmonic n-forms by $\mathcal{H}^r$).

We can check again by simple algebraic manipulations that all the operators defined so far satisfy certain good properties: The Laplacians are self-adjoint operators. Moreover for a (p,q)-form ψ we have $\Delta\psi = 0$ if and only if $\bar{\vartheta}\psi = \bar{\vartheta}^*\psi = 0$.

A very important property of the Laplacians is that they are **elliptic differential operators** of order 2 on the compact manifold X (cf. [8, Chapter 5]).

An application of a general theorem about elliptic differential operators on compact manifolds ([9, Theorem 5.22]) to the Laplacian Δ yields the following result, which we call the **Hodge Theorem**.

Theorem 5.1.4. (Hodge Theorem, cf. [8, p. 84]) 1. $\mathcal{H}^{p,q}$ is a finite-dimensional complex vector space.
2. There is an orthogonal decomposition

$$A_X^{p,q} = \mathcal{H}^{p,q} \oplus \Delta(A_X^{p,q})$$

From this Theorem we can draw the following Corollary.

Corollary 5.1.5. (Hodge decomposition on forms) There exist orthogonal decompositions

$$Z_X^{p,q} = \mathcal{H}^{p,q} \oplus \bar{\vartheta} A_X^{p,q-1}$$
$$A_X^{p,q} = \mathcal{H}^{p,q} \oplus \bar{\vartheta} A_X^{p,q-1} \oplus \bar{\vartheta}^* A_X^{p,q+1}$$

Proof Let $\psi \in \mathcal{H}^{p,q}$. For $\eta \in A_X^{p,q-1}$ we have $(\psi, \bar{\vartheta}\eta) = (\bar{\vartheta}^*\psi, \eta) = (0, \eta) = 0$. Similarly, for $\eta \in A_X^{p,q+1}$, $(\psi, \bar{\vartheta}^*\eta) = (\bar{\vartheta}\psi, \eta) = (0, \eta) = 0$. Hence, by the Hodge Theorem we obtain

$$\bar{\vartheta} A_X^{p,q-1} + \bar{\vartheta}^* A_X^{p,q+1} \subseteq \Delta(A_X^{p,q})$$

But for any $\psi \in \mathcal{H}^{p,q}$ we have $\Delta\psi = \bar{\vartheta}(\bar{\vartheta}^*\psi) + \bar{\vartheta}^*(\bar{\vartheta}\psi) \in \bar{\vartheta} A_X^{p,q-1} + \bar{\vartheta}^* A_X^{p,q+1}$, so in fact

$$\bar{\vartheta} A_X^{p,q-1} + \bar{\vartheta}^* A_X^{p,q+1} = \Delta(A_X^{p,q}) \quad (\dagger)$$

Now for $\psi \in A_X^{p,q-1}, \eta \in A_X^{p,q+1}$ we have $(\bar{\vartheta}\psi, \bar{\vartheta}^*\eta) = (\bar{\vartheta}^2\psi, \eta) = (0, \eta) = 0$, so decomposition (†) is orthogonal as desired.
Finally, for any $\bar{\vartheta}^*\psi \in \bar{\vartheta}^* A_X^{p,q+1}$ which is $\bar{\vartheta}$-closed, we have $0 = (\bar{\vartheta}\bar{\vartheta}^*\psi, \psi) = (\bar{\vartheta}^*\psi, \bar{\vartheta}^*\psi)$ and hence $\bar{\vartheta}^*\psi = 0$. This immediately implies the orthogonal decomposition $Z_X^{p,q} = \mathcal{H}^{p,q} \oplus \bar{\vartheta} A_X^{p,q-1}$ and the proof is complete. □

Remark We presented this proof mainly as an example of the general reasoning in the proofs of most of the results we have stated without proof.

Theorem 5.1.6. We have an isomorphism

$$H^q(X, \Omega_X^p) \cong \mathcal{H}^{p,q}$$

Proof Using Dolbeault's Theorem and the previous Corollary, we obtain

$$H^q(X, \Omega_X^p) \cong H_{\bar{\vartheta}}^{p,q}(X) = Z_X^{p,q} / \bar{\vartheta} A_X^{p,q-1} \cong \mathcal{H}^{p,q}$$

So we are done. □

Remarks 1. We see straight away that $H^q(X, \Omega_X^p)$ are finite dimensional complex vector spaces. We already know this when X is a smooth projective variety by applying GAGA (Statement 3.3.3./1) and recalling Theorem 2.3.1./1.
2. The natural analogues of Theorem 5.1.4. and Corollary 5.1.5. for the other two operators are true in exactly the same fashion.
3. All the above results hold true in more generality, namely if we allow our sheaves to have coefficients in any holomorphic vector bundle E on X endowed with a Hermitian metric. We can repeat the whole procedure and constructions with the operator $\bar{\vartheta}_E$. For more details, see [9, p. 122].

5.2 The Hodge decomposition for compact, Kähler manifolds

We will now see how to obtain a nice decomposition of the singular complex cohomology groups of X as well as how to define an integral Hodge structure of weight r on the integral cohomology group $H^r(X,\mathbb{Z})$, when X is a compact, Kähler manifold.

Let X be compact and Kähler, with ω its Kähler $(1,1)$-form, coming from a Hermitian metric ds^2. Define the operator $L:\mathcal{A}_X^{p,q}\longrightarrow\mathcal{A}_X^{p+1,q+1}$ by $\eta\mapsto\eta\wedge\omega$. Also let $\Lambda:\mathcal{A}_X^{p,q}\longrightarrow\mathcal{A}_X^{p-1,q-1}$ be the operator $\Lambda=L^*=(-1)^{p+q}*L*$. We may check that Λ is the adjoint operator of L with respect to the L^2 inner product defined in the preceding section.

We have the following important identities.

Theorem 5.2.1. (Kähler identities) If $[A,B]=AB-BA$ denotes the commutator of A and B, then

$$[\Lambda,\bar{\vartheta}]=-i\vartheta^*,\ [\Lambda,\vartheta]=i\bar{\vartheta}^*$$

Proof Since we know that ω is a real form, so is the operator Λ and therefore one identity is true if and only if the other is true.
To prove the first identity, we first prove it for compactly supported forms on $\mathbb{C}^n$ with the standard Euclidean metric. Then we use the fact that the Kähler metric on X osculates to the standard Euclidean metric to order 2 to deduce the identity at every point. For more details, see [8, pp 111-114]. □

With these two identities at hand, we can show the following fact about the different Laplacians defined on X.

Theorem 5.2.2. Let X be a compact, Kähler manifold. Then

$$\Delta_\vartheta=\Delta_{\bar{\vartheta}}=\frac{1}{2}\Delta_d$$

Proof This is again just a simple formal algebraic manipulation. We just substitute the adjoints of $\vartheta,\bar{\vartheta}$ in the Laplacians using the Kähler identities.
Since $\Lambda\vartheta-\vartheta\Lambda=i\bar{\vartheta}^*$, we get

$$i(\vartheta\bar{\vartheta}^*+\bar{\vartheta}^*\vartheta)=\vartheta\Lambda\vartheta-\vartheta^2\Lambda+\Lambda\vartheta^2-\vartheta\Lambda\vartheta=0$$

Hence also by conjugation $\bar{\vartheta}\vartheta^*+\vartheta^*\bar{\vartheta}=0$ and so

$$\Delta_d=\Delta_\vartheta+\Delta_{\bar{\vartheta}}+\vartheta\bar{\vartheta}^*+\bar{\vartheta}^*\vartheta+\bar{\vartheta}\vartheta^*+\vartheta^*\bar{\vartheta}=\Delta_\vartheta+\Delta_{\bar{\vartheta}}$$

Now we have

$$-i\Delta_\vartheta=\vartheta(\Lambda\bar{\vartheta}-\bar{\vartheta}\Lambda)+(\Lambda\bar{\vartheta}-\bar{\vartheta}\Lambda)\vartheta=\vartheta\Lambda\bar{\vartheta}-\vartheta\bar{\vartheta}\Lambda+\Lambda\bar{\vartheta}\vartheta-\bar{\vartheta}\Lambda\vartheta$$

$$i\Delta_{\bar{\vartheta}}=\bar{\vartheta}(\Lambda\vartheta-\vartheta\Lambda)+(\Lambda\vartheta-\vartheta\Lambda)\bar{\vartheta}=\bar{\vartheta}\Lambda\vartheta-\bar{\vartheta}\vartheta\Lambda+\Lambda\vartheta\bar{\vartheta}-\vartheta\Lambda\bar{\vartheta}$$

We deduce that $\Delta_\vartheta=\Delta_{\bar{\vartheta}}=\frac{1}{2}\Delta_d$, which concludes the proof. □

This theorem has many immediate but important corollaries, which will eventually lead to the Hodge decomposition. A first point to make is that we do not need to specify which Laplacian we are referring to when we treat harmonic forms. In the rest of this section we will be referring to the operator d unless stated otherwise.

Corollary 5.2.3. If X is Kähler, Δ_d preserves bidegree, i.e. $\Delta_d(A_X^{p,q})\subseteq A_X^{p,q}$. Alternatively, we say that Δ_d is bihomogeneous.

Proof $\Delta_d = 2\Delta_{\bar{\vartheta}}$ and $\Delta_{\bar{\vartheta}}$ is clearly bihomogeneous. $\square$

This also obviously implies the next two statements.

Corollary 5.2.4. The components $\psi^{p,q}$ of a harmonic form $\psi \in A^r_X$ are harmonic.

Corollary 5.2.5. If $\mathcal{H}^{p,q}$ is the set of (p,q)-forms which are d-harmonic, we have a direct sum decomposition

$$\mathcal{H}^r = \bigoplus_{p+q=r} \mathcal{H}^{p,q} \quad (*)$$

From this decomposition we will extract the Hodge decomposition.
By the analog of the Hodge Theorem 5.1.4. for the operator d we deduce that there is an isomorphism

$$H^r_{dR}(X) \cong \mathcal{H}^r \quad (1)$$

Let $H^{p,q}(X)$ be the set of classes of $(p+q)$-forms that can be represented by a closed form of type (p,q).
We show now that we also have an isomorphism

$$H^{p,q}(X) \cong \mathcal{H}^{p,q} \quad (2)$$

Clearly $\mathcal{H}^{p,q} \subseteq H^{p,q}(X)$ by our work so far. So we need to show the reverse inclusion. Let ω be a closed form of type (p,q). By Theorems 5.1.4./2 and 5.2.2. we may uniquely write $\omega = \alpha + \Delta\beta$, where α is harmonic, hence closed. Then $\Delta\beta = dd^*\beta + d^*d\beta$ must be closed. Hence $dd^*d\beta = 0$ which in turn implies that $0 = (dd^*d\beta, d\beta) = (d^*d\beta, d^*d\beta)$ and so $d^*d\beta = 0$. Therefore $\omega = \alpha + dd^*\beta$ and ω, α represent the same class in $H^{p,q}(X)$. Therefore $H^{p,q}(X) \subseteq \mathcal{H}^{p,q}$. So we are done and we have shown that (2) is true.

Finally, notice that since $\Delta_\vartheta = \Delta_{\bar{\vartheta}}$ we get $\mathcal{H}^{q,p} = \overline{\mathcal{H}^{p,q}}$ (3).
Combining all the above ($(*), (1), (2), (3)$) together with the standard smooth de Rham theorem, which gives that $H^r(X, \mathbb{C}) \cong H^r_{dR}(X)$, we obtain the celebrated Hodge decomposition, which was the one we were aiming for.

Theorem 5.2.6. (Hodge decomposition) If X is a compact, Kähler manifold, we have the following decomposition for the complex cohomology

$$H^r(X, \mathbb{C}) \cong \bigoplus_{p+q=r} H^{p,q}(X)$$

$$H^{q,p}(X) = \overline{H^{p,q}(X)}$$

In addition, $\Delta_d = 2\Delta_{\bar{\vartheta}}$ gives that $\mathcal{H}^{p,q}_d = \mathcal{H}^{p,q}_{\bar{\vartheta}}$ and therefore by Theorem 5.1.6. $H^{p,q}(X) \cong H^q(X, \Omega^p_X)$, which gives rise to the decomposition

$$H^r(X, \mathbb{C}) \cong \bigoplus_{p+q=r} H^q(X, \Omega^p_X)$$

Remark Even though we started with a specific Kähler metric on X, we note that the Hodge decomposition which we obtained does not in fact depend on the choice of Kähler metric, as is clear by the definition of the pieces $H^{p,q}(X)$.

Set now $b_r = \dim_{\mathbb{C}} H^r(X, \mathbb{C})$, the r-th Betti number, and $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}(X) = \dim_{\mathbb{C}} H^q(X, \Omega_X^p)$, the Hodge numbers.
Then, noting that $* : \mathcal{H}^{p,q} \longrightarrow \mathcal{H}^{n-p,n-q}$ is a conjugate linear isomorphism, we conclude that these quantities satisfy the relations

$$h^{p,q} = h^{q,p}, \ h^{p,q} = h^{n-p,n-q}, \ b_r = \sum_{p+q=r} h^{p,q}$$

We now present two simple applications of the Hodge decomposition.

Proposition 5.2.7. The Betti numbers b_{2r+1} of odd degree are even.

Proof $b_{2r+1} = \sum_{j \le r} h^{j,2r+1-j} + \sum_{j \ge r+1} h^{j,2r+1-j} = \sum_{j \le r} h^{j,2r+1-j} + \sum_{j \ge r+1} h^{2r+1-j,j} = 2[\sum_{j \le r} h^{j,2r+1-j}]$ and hence b_{2r+1} is even, as desired. □

Proposition 5.2.8. Let $X = \mathbb{P}^n(\mathbb{C})$. Then $H^q(X, \Omega_X^p) = H_{\bar{\partial}}^{p,q}(X)$ equals $\mathbb{C}$ if $p = q$ and 0 if $p \neq q$.

Proof It is a standard fact (recall e.g. the standard decomposition of $\mathbb{P}^n(\mathbb{C})$ as a cell complex) that $H^{2r+1}(X, \mathbb{Z}) = 0$ and $H^{2r}(X, \mathbb{Z}) = \mathbb{Z}$. Since by the universal coefficient theorem we have $H^r(X, \mathbb{C}) = H^r(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$, we get that $b_{2r+1} = 0, \ b_{2r} = 0$. Hence $H_{\bar{\partial}}^{p,q}(X) = 0$ for $p+q$ odd. Moreover

$$1 = b_{2r} = h^{r,r} + 2 \sum_{j<r} h^{j,2r-j}$$

Therefore we must have $\sum_{j<r} h^{j,2r-j} = 0 \Rightarrow h^{j,2r-j} = 0$ for all $j < r$ and $h^{r,r} = 1$. Thus $h^{p,q} = 0$ for $p \neq q$. So we conclude that $H_{\bar{\partial}}^{p,q}(X) = 0$ for $p \neq q$ and $H_{\bar{\partial}}^{p,p}(X) \cong H^{2p}(X, \mathbb{C}) \cong \mathbb{C}$ and we are done. □

5.3 The Hodge filtration and Hodge to de Rham spectral sequence

We first define the notion of a filtered complex and then give a brief account of the basic definitions and properties of spectral sequences associated to a filtered complex.

Definition 5.3.1. Let $(A^\bullet, d)$ be a complex of abelian groups or sheaves of abelian groups supported in non-negative degree. We define a decreasing **filtration** on $A^\bullet$ to be a family of subcomplexes

$$... \hookrightarrow F^p A^\bullet \hookrightarrow F^{p-1} A^\bullet \hookrightarrow ... \hookrightarrow F^0 A^\bullet = A^\bullet$$

Then we say that $A^\bullet$ together with this filtration is a **filtered complex** $(F^p A^\bullet, d)$.

Definition 5.3.2. (Spectral sequence) A **spectral sequence** is a collection of complexes $(E_r^{p,q}, d_r)$, $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ such that $E_{r+1}^{p,q}$ is identified with the cohomology of $(E_r^{p,q}, d_r)$, i.e. with $\ker(d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}) / \mathrm{im}(d_r : E_r^{p-r,q+r-1} \to E_r^{p,q})$.

Definition 5.3.3. We say that a spectral sequence $(E_r^{p,q}, d_r)$ degenerates at E_r if $\forall k \ge r$ we have $d_k = 0$. Then $E_r^{p,q} = E_\infty^{p,q}$.

In what follows, the filtrations considered will all be finite, i.e. $F^p A^\bullet = 0$ for p sufficiently large. So we assume this is the case throughout, i.e. we always deal with finite filtrations.
Now it is easy to see that a filtration naturally induces a filtration $F^p H^*(A^\bullet)$ on the cohomology of $A^\bullet$ given by $F^p H^i(A^\bullet) = \mathrm{im}(H^i(F^p A^\bullet) \to H^i(A^\bullet))$. We set

$$\mathrm{Gr}^p A^\bullet = F^p A^\bullet / F^{p+1} A^\bullet$$

$$\mathrm{Gr}^p H^q(A^\bullet) = F^p H^q(A^\bullet)/F^{p+1} H^q(A^\bullet)$$

We have now the following important foundational result about the spectral sequence associated to a filtered complex.

Theorem 5.3.4. ([9, Theorem 8.21] & [9, Lemma 8.24]) Let $(F^p A^\bullet, d)$ be a filtered complex. Then there exists a spectral sequence $(E_r^{p,q}, d_r)$ with

$$E_0^{p,q} = \mathrm{Gr}^p A^{p+q}$$

$$E_1^{p,q} = H^{p+q}(\mathrm{Gr}^p A^\bullet)$$

$$E_\infty^{p,q} = \mathrm{Gr}^p(H^{p+q}(A^\bullet))$$

Moreover, the differential $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ can be identified with the connection map $\delta : H^{p+q}(\mathrm{Gr}^p A^\bullet) \to H^{p+q+1}(\mathrm{Gr}^{p+1} A^\bullet)$ which appears in the long exact sequence associated to the short exact sequence

$$0 \longrightarrow \mathrm{Gr}^{p+1} A^\bullet \longrightarrow F^p A^\bullet / F^{p+2} A^\bullet \longrightarrow \mathrm{Gr}^p A^\bullet \longrightarrow 0$$

We now give two types of filtration on complexes, which will be the main examples we will be concerned with.

Examples 5.3.5. 1. (The "naive" filtration) Let $(A^\bullet, d)$ be a complex of abelian groups. We set $F^p A^\bullet = A^{\geq p}$. This is the complex which is zero in degrees smaller than p and equals $A^\bullet$ in degrees $\geq p$.

$$\begin{array}{ccccccccccc} A^0 & \xrightarrow{d} & A^1 & \xrightarrow{d} & \dots & \xrightarrow{d} & A^p & \xrightarrow{d} & A^{p+1} & \xrightarrow{d} & \dots \\ \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & \dots & \longrightarrow & A^p & \xrightarrow{d} & A^{p+1} & \xrightarrow{d} & \dots \end{array}$$

2. (The filtration of a double complex) Let $(A^{\bullet,\bullet}, d, \delta)$ be a double complex supported in non-negative degree. Let $(A^\bullet, D)$ be the associated simple complex. Recall that this is given by $A^r = \bigoplus_{p+q=r} A^{p,q}$, $D = d + \delta$. Set

$$F^p A^r = \bigoplus_{p'+q=r,\ p'\geq p} A^{p',q}$$

It is easy to see that this indeed gives a filtration of the complex $(A^\bullet, D)$.

In the case described in the last example, Theorem 5.3.3. simplifies and we have the following Proposition.

Proposition 5.3.6. Let $(A^\bullet, D)$ be the simple complex associated to the the double complex $(A^{\bullet,\bullet}, d, \delta)$. If $'F^p A^r = \oplus_{p'+q=r,\ p'\geq p} A^{p',q}$ is the filtration introduced in Example 5.3.5./2, the spectral sequence $'E_r^{p,q}$ associated to $A^\bullet$ satisfies the following:
1. $'E_0^{p,q} = A^{p,q}$, $d_0 = \delta$.
2. $'E_1^{p,q} = H^q(A^{p,\bullet})$, and the differential $d_1 : H^q(A^{p,\bullet}) \to H^q(A^{p+1,\bullet})$ is induced by the morphism of complexes $d : A^{p,\bullet} \to A^{p+1,\bullet}$.
3. $'E_2^{p,q} = H^*('E_1^{p,q}, d_1) = H_d^p(H_\delta^q(A^{\bullet,\bullet}))$, where this denotes the cohomology of

$$\dots \longrightarrow H_\delta^q(A^{p-1,\bullet}) \xrightarrow{d} H_\delta^q(A^{p,\bullet}) \xrightarrow{d} H_\delta^q(A^{p+1,\bullet}) \longrightarrow \dots$$

Similarly, we have a spectral sequence $''E_r^{p,q}$ associated to the filtration $''F^p A^r = \oplus_{p+q'=r,\ q'\geq p} A^{p,q'}$ which has the corresponding properties.

We will now see how to apply all the above to obtain an analogous result as in the preceding section.

Let X be a complex manifold and $(\Omega_X^\bullet, \vartheta)$ be the holomorphic de Rham complex. This complex has the "naive" filtration given in Example 5.3.5./1 $F^p\Omega_X^\bullet = \Omega_X^{\geq p}$.

We also have a filtration on the de Rham complex of X

$$F^p\mathcal{A}_X^r = \bigoplus_{p'+q=r,\ p'\geq p} \mathcal{A}_X^{p',q}$$

Passing to global sections we obtain the filtration

$$F^pA_X^r = \bigoplus_{p'+q=r,\ p'\geq p} A_X^{p',q} \quad (\dagger)$$

This induces a filtration on cohomology given by

$$F^pH^r(X,\mathbb{C}) = F^pH^r(X,A_X^\bullet) = \ker(F^pA_X^r \to F^pA_X^{r+1})/\mathrm{im}(F^pA_X^{r-1} \to F^pA_X^r) \quad (\dagger\dagger)$$

We know that the complex $(\mathcal{A}_X^{p,\bullet}, \bar{\vartheta})$ is a resolution of Ω_X^p. It is a fact then that $(\mathcal{A}_X^\bullet, d)$, which is the single complex associated to the double complex $(\mathcal{A}_X^{\bullet,\bullet}, \vartheta, \bar{\vartheta})$, together with this filtration is quasi-isomorphic to $(\Omega_X^\bullet, d)$ together with the naive filtration (see [9, Lemma 8.5, p. 201]), the quasi-isomorphism being between these filtered complexes at all levels of the two filtrations. That is,

$$\mathbb{H}^*(X, F^p\mathcal{A}_X^\bullet) \cong \mathbb{H}^*(X, F^p\Omega_X^\bullet) \quad (1)$$

Hence these filtrations are closely related and to obtain information about the latter we will examine the former.

Since the sheaves $\mathcal{A}_X^q$ are fine, hence acyclic, by a known theorem the hypercohomology of $(F^p\mathcal{A}_X^\bullet, d)$ is equal to the cohomology of the complex of its global sections $(F^pA_X^\bullet, d)$ (cf. [9, Proposition 8.12]), i.e.

$$\mathbb{H}^*(X, F^p\mathcal{A}_X^\bullet) \cong H^*(X, F^pA_X^\bullet) \quad (2)$$

(1) & (2) together imply that we need to examine just the complex $(A_X^\bullet, d)$ together with the filtration (†).

Definition 5.3.7. (The Fröhlicher or Hodge to de Rham spectral sequence) The spectral sequence $'E_r^{p,q}$ associated to the filtration F^p on the de Rham complex of $(A_X^\bullet, d)$ given by (†) is called the **Fröhlicher** or **Hodge to de Rham** spectral sequence.

Since $(A_X^\bullet, d)$ is the single complex associated to the double complex $(A_X^{\bullet,\bullet}, \vartheta, \bar{\vartheta})$ Proposition 5.3.6. applies and we have a very good description of the first terms of the Hodge to de Rham spectral sequence.

In particular, $'E_1^{p,q} = H^q(A_X^{p,\bullet}, \bar{\vartheta})$ and by Dolbeault's theorem we get that $H^q(A_X^{p,\bullet}, \bar{\vartheta}) = H_{\bar{\vartheta}}^{p,q}(X) \cong H^q(X, \Omega_X^p)$ and hence $'E_1^{p,q} = H^q(X, \Omega_X^p)$. Moreover, the differential $d_1 : {'E_1^{p,q}} \to {'E_1^{p+1,q}}$ is induced by ϑ and is just the map

$$d = \vartheta : H^q(X, \Omega_X^p) \longrightarrow H^q(X, \Omega_X^{p+1})$$

Suppose now that X is compact Kähler. Then, by the results of the previous section, every element of $H_{\bar{\vartheta}}^{p,q}(X)$ has a $\bar{\vartheta}$-harmonic representative and so, since $\Delta_\vartheta = \Delta_{\bar{\vartheta}}$, a ϑ-harmonic and hence ϑ-closed representative. This implies that $\vartheta = 0$ on $H^{p,q}_{\vartheta}(X) \cong H^q(X, \Omega^p_X)$ and therefore $d_1 = 0$. We thus have shown the following.

Theorem 5.3.8. (Degeneracy of the Hodge to de Rham spectral sequence) Let X be a compact, Kähler manifold. Then the Hodge to de Rham spectral sequence degenerates at the first term $'E_1^{p,q}$ and hence $'E_1^{p,q} = {}'E_\infty^{p,q}$.

Remark We note that it is not hard to see that Theorem 5.3.8. is equivalent to the existence of an isomorphism $\mathrm{Gr}^p H^{p+q}(X) \cong H^q(X, \Omega^p_X)$. Generally, it is clear from the above that the former is isomorphic to a quotient of the latter, which reduces the existence of such an isomorphism to their dimensions being equal. That is, $\dim {}'E_\infty^{p,q} \leq {}'E_1^{p,q}$ with equality if and only if the spectral sequence degenerates at $'E_1$.

Theorem 5.3.8. is important. We can notice immediately that it gives $\mathrm{Gr}^p H^{p+q}(A^\bullet_X) = {}'E_\infty^{p,q} = {}'E_1^{p,q} = H^q(X, \Omega^p_X)$. Hence we obtain $b_r = \sum_{p+q=r} h^{p,q}$, where b_r is the r-th Betti number and $h^{p,q} = \dim_{\mathbb{C}} H^q(X, \Omega^p_X)$. Therefore, Theorem 5.3.8. is reasonably close to the actual Hodge decomposition, as stated in Theorem 5.2.6., missing the conjugacy condition, the symmetry of the Hodge numbers and the decomposition into pieces of the form $H^{p,q}(X) = F^p H^{p+q}(X, \mathbb{C}) \cap \overline{F^q H^{p+q}(X, \mathbb{C})}$ (see below), instead of $H^q(X, \Omega^p_X)$.
At this point we mention that (cf. also [9, Proposition 7.5, p. 158]) the filtration (††) induced by (†) on cohomology coincides with the standard **Hodge filtration** (see below)

$$F^p H^r(X, \mathbb{C}) = \bigoplus_{p'+q=r,\ p' \geq p} H^{p',q}(X) = \bigoplus_{p'+q=r,\ p' \geq p} H^q(X, \Omega^{p'}_X)$$

In order to prove Theorem 5.3.8. we used results from the harmonic theory of compact, Kähler manifolds. It is natural to ask whether there is an alternative way of proving this fact.

This can indeed be done, as our earlier work enables us to make use of algebraic methods. Suppose that X is a smooth projective complex variety (so X^{an} is a compact, Kähler manifold). All the sheaves involved are coherent. Hence GAGA shows that the degeneracy of the Hodge to de Rham spectral sequence of the holomorphic de Rham complex of X^{an} at the first term is equivalent to the degeneracy at the first term of the corresponding spectral sequence $''E_r^{p,q}$ (cf. Section 4.1 for the definition) for the algebraic de Rham complex. So we are reduced to an algebraic computation.

Another nice way of obtaining some of the above in an algebraic manner is to recall what we have seen in our discussion of Grothendieck's algebraic de Rham Theorem. The theorem shows that $H^*(X, \mathbb{C}) = H^*_{dR}(X) = \mathbb{H}^*(X, \Omega^\bullet_X)$, where everything is in the algebraic setting. The degeneracy of $''E_r^{p,q}$ at the first term will thus yield an isomorphism $''E_\infty^{p,q} \cong H^q(X, \Omega^p_X)$, whence similar results among which, also by GAGA, the equality $b_r = \sum_{p+q=r} h^{p,q}$.

To conclude this section, we define in full generality the notion of an integral Hodge structure and the associated Hodge filtration, motivated by our work so far.

Definition 5.3.9. An **integral Hodge structure** of weight r is given by a free abelian group of finite type $V_{\mathbb{Z}}$, together with a decomposition

$$V_{\mathbb{C}} = V_{\mathbb{Z}} \otimes \mathbb{C} = \bigoplus_{p+q=r} V^{p,q}$$

satisfying $V^{p,q} = \overline{V^{q,p}}$.

Given this Hodge decomposition, we can define the associated **Hodge filtration** by

$$F^p V_{\mathbb{C}} = \bigoplus_{p'+q=r,\ p'\geq p} V^{p',q}$$

We can notice that this is a decreasing filtration on $V_{\mathbb{C}}$ and we have $V_{\mathbb{C}} = F^p V_{\mathbb{C}} \oplus \overline{F^{r-p+1} V_{\mathbb{C}}}$. Moreover the filtration determines the decomposition by $V^{p,q} = F^p V_{\mathbb{C}} \cap \overline{F^q V_{\mathbb{C}}}$.

It is clear now that the Hodge decomposition on the singular cohomology (where we consider the integral cohomology modulo torsion if needed) of a compact, Kähler manifold X as given in Theorem 5.2.6. satisfies all these properties and so induces an associated Hodge filtration. These are the motivation behind the above definitions.

5.4 Deformations of complex structures

In this brief section, we will examine how the Hodge numbers and the Hodge decomposition behave under small deformations of the complex structure of a compact, Kähler manifold. We will follow again Voisin [9], mainly by stating the relevant results. The main point will be that the Hodge numbers are preserved and so does the Hodge decomposition.

Let $\mathcal{X}, B$ be complex manifolds and $\phi : \mathcal{X} \longrightarrow B$ a holomorphic map. Let also $X_t := \phi^{-1}(t)$ be the fibre of ϕ above the point $t \in B$.

Definition 5.4.1. (Family of complex manifolds) $\phi : \mathcal{X} \longrightarrow B$ will be called a family of complex manifolds if ϕ is a proper holomorphic submersion.

It is not difficult that under the conditions of the definition, the fibres X_t are complex manifolds, which we call deformations of X_0.
In the following we will assume that there exists a base point $0 \in B$, which will serve as a point of reference. We state the following result about local trivialisations of a family of complex manifolds.

Lemma 5.4.2. Let $\phi : \mathcal{X} \longrightarrow B$ be a family of complex manifolds. Then there exists a neighbourhood $0 \in U \subseteq B$ and a diffeomorphism $T = (T_0, \phi) : \mathcal{X}|_{\phi^{-1}(U)} \longrightarrow X_0 \times U$ such that the fibres of T_0 are complex submanifolds of $\mathcal{X}$.

On the one hand, this result says that the $\mathcal{C}^\infty$ structure on X_t for $t \in U$ is the same, all these manifolds being diffeomorphic via T. So we can locally view a family of complex manifolds equivalently as a deformation of the complex structure on the fixed smooth manifold X_0. Moreover, we see that this family of complex structures on X_0, X_t, parametrised by $t \in U$ varies holomorphically with respect to t.

The following semicontinuity theorem is the basic foundational result for the behaviour of the Hodge numbers and the Hodge decomposition under this type of small deformations.

Theorem 5.4.3. [9, Theorem 9.15, pp. 232-234] Let $\phi : \mathcal{X} \longrightarrow B$ be a family of compact complex manifolds and $\mathcal{F}$ a holomorphic vector bundle on $\mathcal{X}$. Then the function $t \mapsto \dim H^q(X_t, \mathcal{F}|_{X_t})$ is upper semicontinuous, i.e. $\dim H^q(X_t, \mathcal{F}|_{X_t}) \leq \dim H^q(X_0, \mathcal{F}|_{X_0})$ for all t in a neighbourhood of 0.

Corollary 5.4.4. The function $t \mapsto h^{p,q}(X_t) = \dim H^q(X_t, \Omega^p_{X_t})$ is upper semicontinuous.

With these results at hand, we proceed to show that under small deformations of complex structure the Hodge numbers remain constant and so does the Hodge decomposition. From now on we assume that $\phi : \mathcal{X} \longrightarrow B$ is a family of compact complex manifolds and X_0 is Kähler.

Proposition 5.4.5. For t close to $0 \in B$, we have $h^{p,q}(X_t) = h^{p,q}(X_0)$. Also the Hodge to de Rham spectral sequence of X_t degenerates at the first term.

Proof By Corollary 5.4.4. we have $h^{p,q}(X_t) \leq h^{p,q}(X_0)$ for all t sufficiently close to 0. Now, $H^q(X_t, \Omega^p_{X_t}) = E_1^{p,q}(X_t)$, where $E_r^{p,q}(X_t)$ is the Hodge to de Rham spectral sequence of X_t. We know that dim $E_\infty^{p,q}(X_t) \leq$ dim $E_1^{p,q}(X_t)$ with equality if and only if the spectral sequence degenerates at the E_1 (recall the last remark of the previous section). Morever, $E_\infty^{p,q}(X_t) = F^pH^{p+q}(X_t)/F^{p+1}H^{p+q}(X_t)$ which implies the equality dim $H^r(X_t) = \sum_{p+q=r} \dim E_\infty^{p,q}(X_t)$. As we have remarked already, Lemma 5.4.2. implies that X_t is diffeomorphic to X_0 and hence $H^r(X_t, \mathbb{C}) \cong H^r(X_0, \mathbb{C}) \Rightarrow \dim H^r(X_t, \mathbb{C}) = \dim H^r(X_0, \mathbb{C}) =: b_r$. We have now

$$b_r = \sum_{p+q=r} \dim E_\infty^{p,q}(X_t) \leq \sum_{p+q=r} \dim E_1^{p,q}(X_t) = \sum_{p+q=r} \dim H^q(X_t, \Omega^p_{X_t}) =$$

$$= \sum_{p+q=r} h^{p,q}(X_t) \leq \sum_{p+q=r} h^{p,q}(X_0) = b_r$$

where the last equality follows from the Hodge decomposition (5.2.6.). It follows that we must have equality in all the intermediate inequalities and therefore $h^{p,q}(X_t) = h^{p,q}(X_0)$ and also dim $E_\infty^{p,q}(X_t) =$ dim $E_1^{p,q}(X_t)$, which give us what we want. □

Proposition 5.4.6. For t close to $0 \in B$, there exists a decomposition $H^r(X_t, \mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}(X_t)$, where $H^{p,q}(X_t) = \overline{H^{q,p}(X_t)}$ and $H^{p,q}(X_t) \cong H^q(X_t, \Omega^p_{X_t})$.

Proof By the previous Proposition, we get that the subspace $F^pH^r(X_t, \mathbb{C}) \subseteq H^r(X_t, \mathbb{C}) \cong H^r(X_0, \mathbb{C})$ is of dimension independent of t close to 0 and also varies in a $\mathcal{C}^\infty$ way by Theorem 5.4.7. below. Set $H^{p,q}(X_t) = F^pH^r(X_t, \mathbb{C}) \cap \overline{F^qH^r(X_t, \mathbb{C})}$. We obtain that the dimension of $H^{p,q}(X_t)$ is equal to that of $H^{p,q}(X_0)$, i.e.

$$\dim H^{p,q}(X_t) = \dim H^{p,q}(X_0) = h^{p,q}(X_0) = h^{p,q}(X_t) = \dim H^q(X_t, \Omega^p_{X_t}) \quad (1)$$

By the Hodge decomposition for X_0 we have

$$H^r(X_0, \mathbb{C}) = F^pH^r(X_0, \mathbb{C}) \oplus \overline{F^{q+1}H^r(X_0, \mathbb{C})} \quad (2)$$

where $r = p + q$. By continuity, the same holds for t close to 0. This decomposition for X_t implies that the map $H^{p,q}(X_t) \hookrightarrow F^pH^r(X_t, \mathbb{C}) \to F^pH^r(X_t, \mathbb{C})/F^{p+1}H^r(X_t, \mathbb{C}) \cong H^q(X_t, \Omega^p_{X_t})$ is injective, where the last isomorphism is again due to the previous Proposition. So by (1) we get that this map is in fact an isomorphism, whence $H^{p,q}(X_t) \cong H^q(X_t, \Omega^p_{X_t})$.
Finally, (2) for t together with the rest of the above give exactly the decomposition

$$H^r(X_t, \mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}(X_t)$$

and also it is clear again by continuity that $H^{p,q}(X_t) = \overline{H^{q,p}(X_t)}$. □

Remark In fact, even more is true: Kodaira showed that if X_0 is Kähler, then so are all the fibres X_t for t in a neighbourhood of $0 \in B$.

To finish this section and for the sake of completeness, we state a result by Kodaira, which is the basis for Theorem 5.4.3., and use it to give a proof.

Theorem 5.4.7. [7, Kodaira (1986)] Let $\phi : \mathcal{X} \longrightarrow B$ be a family of compact complex manifolds and $G \longrightarrow \mathcal{X}$ a vector bundle. Let Δ be a relative differential operator acting on G, that is $\Delta_t = \Delta_{X_t} : G_{X_t} \longrightarrow G_{X_t}$ is a differential operator. If each Δ_t is elliptic of fixed order, then $t \mapsto \dim \ker\Delta_t$ is upper semicontinuous, and $\ker\Delta_t$ varies in a $\mathcal{C}^\infty$ way and forms a complex subbundle of G.

Proof of Theorem 5.4.3. Endow $\mathcal{X}$ and $\mathcal{F}$ with Hermitian metrics. These induce Hermitian metrics on X_t and $\mathcal{F}|_{X_t}$. Now we apply Theorem 5.4.7. to the $\bar{\vartheta}-$Laplacian Δ_t that acts on the sections of $\mathcal{A}^{0,q}_{X_t}(F_t)$, where F_t is the holomorphic vector bundle associated to $\mathcal{F}|_{X_t}$. But by the generalised version of Theorem 5.1.6. (recall the remarks at the end of the Section 5.1) we have that $\ker\Delta_t = \mathcal{H}^{0,q}_{\bar{\vartheta}}(F_t) \cong H^q(X_t, \mathcal{F}|_{X_t})$ and so we are done. $\square$

References

[1] Serre J-P. *Géométrie Algébrique et Géométrie Analytique*, Annales de l' institut Fourier, tome 6 (1956), pp. 1-42

[2] Serre J-P. *Faisceaux Algébriques Cohérents*, Annals of Mathematics, Vol. 61, No. 2 (1955), pp. 197-278

[3] Grothendieck A. *On the de Rham cohomology of algebraic varieties*, Publications mathématiques de l' I.H.É.S., tome 29 (1966), pp. 95-103

[4] Deligne P. *Théorie de Hodge: II*, Publications mathématiques de l' I.H.É.S., tome 40 (1971), pp. 5-57

[5] Deligne P. *Théorie de Hodge: III*, Publications mathématiques de l' I.H.É.S., tome 44 (1974), pp. 5-77

[6] Deligne P. *Travaux de Griffiths*, Séminaire Bourbaki, exp. 376 (1969-1970), pp. 213-237

[7] Kodaira K. *Complex Manifolds and Deformation of Complex Structures*, Grundlehren der mathematischen Wissenschaften 283, Springer (1986)

[8] Griffiths P. & Harris J. *Principles of Algebraic Geometry*, Wiley & Sons (1978)

[9] Voisin C. *Hodge Theory and Complex Algebraic Geometry, I*, Cambridge studies in advanced mathematics, Cambridge University Press (2002)

[10] Voisin C. *On the Cohomology of Algebraic Varieties*, Proceedings of the International Congress of Mathematicians (Hyderabad, India, 2010), Vol. I, pp. 476-503

[11] Hartshorne R. *Algebraic Geometry*, Graduate Texts in Mathematics 52, Springer (1977)

[12] Kempf G. *Algebraic Varieties*, London Mathematical Society Lecture Note Series 172, Cambridge University Press (1993)

[13] Vakil R. *Foundations of Algebraic Geometry*, unpublished notes available online at `http://math.stanford.edu/ vakil/216blog/`

[14] Pottharst J. *Basics of Algebraic de Rham Cohomology*, unpublished notes available online at `http://www.math.harvard.edu/ jay/writings/derham.pdf`