MATH 102 Fall 2019 – Final Exam Study Guide

The following provides a list of concepts that you should be familiar with for the final exam. It consists of essential points we have covered as well as some examples that appeared in lecture or in the homework together with occasional comments. The exam is cumulative and will cover all the material we have covered up to Section 6.5. You should also refer to the textbook, lecture notes and homework for an idea of what you may be expected to know. Reviewing examples done in lecture and assigned as homework will provide you with a solid understanding of concepts which may be tested. However, you may be asked to apply understanding of these concepts in new ways on the exam, so it is important that you master the underlying concepts and fully understand the motivation of each step of the solution in addition to knowing how to solve the exercises you review.

Section 3.5 - Change of Basis

- Coordinate vector $[v]_E \in \mathbb{R}^n$ of an element $v$ of a vector space $V$ with respect to a given basis $E$.
- Transition matrix between two ordered bases for a vector space.
- Finding coordinate vectors and transition matrices in examples ($\mathbb{R}^n$, vector spaces of polynomials etc.).

Section 4.1 - Linear Transformations

- Definition and properties of linear transformations $L: V \rightarrow W$.
- Kernel, image of a subspace and range of a linear transformation.
- Linear transformation $L_A: \mathbb{R}^n \rightarrow \mathbb{R}^m$ arising from an $m \times n$ matrix $A$.
- Verifying whether a mapping $L: V \rightarrow W$ is linear.
- For a linear transformation $L: V \rightarrow W$, finding its kernel, range and $L(S)$ for a subspace $S \subseteq V$ and proving facts about them.

Section 4.2 - Matrix Representations

- Finding matrix representing a linear transformation $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$.
- Finding matrix representing a linear transformation $L: V \rightarrow W$ with respect to ordered bases $E$ for $V$ and $F$ for $W$.
- 2D graphics and homogeneous coordinates. Understanding how to represent a 2D figure with $n$ vertices in homogeneous coordinates via a $3 \times n$ matrix with a bottom row of ones.
• Geometric transformations in \( \mathbb{R}^2 \) and how to apply them: dilations/contractions, reflection about the \( x \)-axis or \( y \)-axis, rotations and the matrices representing them in standard or homogeneous coordinates. Translations and the matrices representing them in homogeneous coordinates.

Section 4.3 - Similarity

• Definition of similarity of matrices.
• Matrices representing a linear operator with respect to two different bases are similar, using the transition matrix between the two bases.
• Proving facts about similar matrices.

Section 5.1 - Scalar Product in \( \mathbb{R}^n \)

• Scalar product \( x^T y \) in \( \mathbb{R}^n \).
• Length \( ||x|| \) of a vector, angle between two vectors, unit vector \( u = \frac{1}{||x||}x \) in the direction of \( x \).
• Cauchy-Schwarz inequality.
• Orthogonality and Pythagorean Law.
• Scalar and vector projections.
• Planes in \( \mathbb{R}^3 \) and their equation using a normal vector \( N \).
• Finding the distance from a point to a plane using orthogonal projection.
• Understanding and using the algorithm for finding the best match in a database search.
• Computing the correlation matrix between different sets of data and understanding what it means.

Section 5.2 - Orthogonal Subspaces

• Orthogonal subspaces, orthogonal complements and their properties.
• Fundamental subspaces of a matrix \( A \) and Fundamental Subspaces Theorem.
• Direct sums of subspaces.
• \( \mathbb{R}^n = S \oplus S^\perp \), where \( S \) is a subspace of \( \mathbb{R}^n \).
• Computing orthogonal complements and bases for fundamental subspaces in examples.
• Proving facts about fundamental subspaces and direct sums.
Section 5.3 - Least Squares Problems

- Process of finding least squares solution to an overdetermined system $Ax = b$ by projecting $b$ onto $R(A)$ to minimize the norm $||r(x)||$ of the residual vector $r(x) = b - Ax$.

- Normal equations $A^TAx = A^Tb$. Solution $\hat{x} = (A^TA)^{-1}A^Tb$, projection matrix $P = A(A^TA)^{-1}A^T$ and projection vector $p = A(A^TA)^{-1}A^Tb$ of $b$ onto $R(A)$, when $A$ has rank $n$ (equal to the number of its columns).

- Finding least squares solutions to linear systems.

- Using least squares to fit data using a certain model (linear, quadratic, cubic etc.).

- Proving facts about projection matrices and least squares.

Section 5.4 - Inner Product Spaces

- Definition of inner product space and examples (scalar product on $\mathbb{R}^n$, inner products on spaces of functions).

- Length/norm of a vector, orthogonality, Pythagorean law.

- Scalar and vector projection of a vector $u$ onto a vector $v$.

- Cauchy-Schwarz inequality, angle between two vectors.

- Normed linear spaces and examples (norms from inner products, $p$-norm on $\mathbb{R}^n$).

- Distance between two vectors.

- Proving facts about inner products and norms.

Section 5.5 - Orthonormal Sets

- Orthogonal and orthonormal sets in an inner product space and examples ($\mathbb{R}^n$, “Fourier” orthonormal set).

- Properties of orthonormal sets, Parseval’s formula.

- Projection onto a subspace $S$ of $V$ using an orthonormal basis for $S$.

- Proving facts about orthonormal sets and projections.

- Approximating periodic functions by trigonometric polynomials.

- Approximating functions by linear functions.

Section 5.6 - Gram-Schmidt Process

- Gram-Schmidt Process and examples ($\mathbb{R}^n$, orthogonal polynomials).

- QR factorization of a matrix.

- Using Gram-Schmidt in examples and to prove facts about inner product spaces.
Section 6.1 - Eigenvalues and Eigenvectors

- Eigenvalues, eigenvectors, eigenspaces, characteristic polynomial of a matrix.
- Sum and product of eigenvalues are the trace and determinant of a matrix respectively.
- Similar matrices have the same characteristic polynomial and eigenvalues.
- Computing eigenvalues and eigenvectors.
- Proving facts about eigenvalues and eigenvectors.

Section 6.2 - Systems of Linear Differential Equations

- Solving first and second order systems of linear differential equations with constant coefficients and initial value problems.
- Setting up and solving problems modelling liquid mixtures in tanks using systems of differential equations.

Section 6.3 - Diagonalization

- Definition of diagonalizable and defective matrices.
- An $n \times n$ matrix is diagonalizable if and only if it has $n$ linearly independent eigenvectors.
- Matrices with distinct eigenvalues are diagonalizable.
- Diagonalizing square matrices in examples.
- Proving facts about diagonalizable matrices.
- Definition of stochastic matrices and probability vectors.
- Markov processes, transition matrix, steady-state vector.
- Finding the steady-state vector for a Markov process.

Section 6.5 - Singular Value Decomposition

- Singular Value Decomposition of a matrix.
- Properties of the Singular Value Decomposition and the singular vectors.
- Computing the Singular Value Decomposition of a matrix in examples.
- Approximating a matrix by a matrix of lower rank with respect to the Frobenius norm using its Singular Value Decomposition.
- Proving simple facts about or by using the Singular Value Decomposition.