Math 102 HW#6
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The following are solutions to Section 5.5:28, 29, 32; Section 5.6: 3(b), 5 (a)-(b), 7, 8, 14; Section 6.1: 1 (a), (b), (k), 2, 3, 4

Problem 28. Let $p$ be the projection of a vector $v$ onto an subspace. Show that $\|p\|^2 = \langle p, v \rangle$.

Proof. By definition of projection, $p \in S$, $(v - p) \in S^\perp$, thus $\langle v - p, p \rangle = 0$.

$$\langle p, v \rangle = \langle p, p \rangle + \langle p, v - p \rangle$$

\[ \square \]

Problem 29. Define the inner product and induced norm on $C[-1,1]$  

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx, \quad \|f\| = \sqrt{\langle f, f \rangle}$$

1. Show that $1$ and $x$ are orthogonal.
2. Compute $\|1\|$ and $\|x\|$.
3. Find the best least squares approximation to $x^{1/3}$ on $[-1, 1]$ by a linear function $l(x) = c_1 1 + c_2 x$.
4. Sketch the graphs of $x^{1/3}$ and $l(x)$ on $[-1, 1]$.

Proof. 1. Verify by definition and use the fact that $x$ is a odd function.

2. $\|1\| = \sqrt{2}$, $\|x\| = \sqrt{\frac{2}{3}}$.

3. Compute the projection of $x^{1/3}$ on the subspace $span\{1, x\}$. With the fact that $1$ and $x$ are orthogonal

$$\|1\|c_1 = \langle 1, x^{1/3} \rangle = 0, \quad \|x\|c_2 = \langle x, x^{1/3} \rangle = \frac{9}{7}$$

$$\implies c_1 = 0, c_2 = \frac{6}{7}$$

\[ \square \]

Problem 32. Find the best least squares approximation to $f(x) = |x|$ on $[-\pi, \pi]$ by a trigonometric polynomial of degree less than or equal to 2.

Proof. Let the approximation be

$$t_2 = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

where

$$a_0 = \langle |x|, 1 \rangle = \pi, \quad a_1 = \langle |x|, \cos x \rangle = \frac{4}{\pi}, \quad a_2 = \langle |x|, \cos 2x \rangle = 0$$

$b_1 = b_2 = 0$

\[ \square \]
Problem 5. Let
\[ A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 12 \\ 6 \\ 18 \end{bmatrix} \]

1. Use the Gram-Schmidt process to find an orthonormal basis for the column space of \( A \).
2. Factor \( A \) into a product \( QR \).

**Proof.** Denote \( A = [a_1, a_2] \) and let \( q_1, q_2 \) be an orthonormal basis for \( Col(A) \).

\[
\|a_1\| = \sqrt{4+1+4} = 3, \quad q_1 = \frac{1}{3}(2, 1, 2)^T \\
\langle a_2, q_1 \rangle = \frac{5}{3} a_2 - \frac{5}{3} q_1 = -\frac{1}{9}(1, -4, 1)^T, \quad q_2 = -\frac{\sqrt{2}}{6}(1, -4, 1)^T
\]

\( \square \)

Problem 7. Given \( x_1 = \frac{1}{2}(1, 1, 1, -1)^T \) and \( x_2 = \frac{1}{2}(1, 1, 3, 5)^T \), verify that they are orthonormal and extend it to be an orthonormal set in \( \mathbb{R}^4 \).

**Proof.** Here I only provide the process of completing a full orthonormal set in \( \mathbb{R}^4 \). By solving the linear system \( Ax = 0 \) for
\[
A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 3 & 5 \end{bmatrix}
\]
We get a basis for \( N(A) \): \( B = \{b_1, b_2\} \) where
\[
b_1 = (-1, 1, 0, 0)^T, \quad b_2 = (4, 0, -3, 1)^T
\]
Through Gram-Schmidt process, we can transform \( B \) into an orthonormal basis: \( \{q_3, q_4\} \) where
\[
q_3 = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0, 0)^T, \quad q_4 = \frac{1}{3\sqrt{2}}(2, 2, -3, 1)^T
\]

\( \square \)

Problem 8. Use the Gram-Schmidt process to find an orthonormal basis for the subspace of \( \mathbb{R}^4 \) spanned by \( x_1 = (4, 2, 2, 1)^T, x_2 = (2, 0, 0, 2)^T, \) and \( x_3 = (1, 1, -1, 1)^T \).

**Proof.** It’s just about computation, the results should be
\[
q_1 = \frac{1}{5}(4, 2, 2, 1)^T, \quad q_2 = \frac{1}{5}(1, -2, -2, 4)^T, \quad q_3 = \frac{\sqrt{2}}{2}(0, 1, -1, 0)^T
\]

\( \square \)

Problem 14. Let \( U \) be an \( m \)-dimensional subspace of \( \mathbb{R}^n \) and let \( V \) be a \( k \)-dimensional subspace of \( U \), where \( 0 < k < m \).

1. Show that any orthonormal basis \( \{v_1, v_2, \ldots, v_k\} \) for \( V \) can be expanded to form an orthonormal basis \( \{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_m\} \) for \( U \).
2. Show that if \( W = \text{Span}(v_1, v_2, \ldots, v_k) \), then \( U = V \oplus W \).

**Proof.** 1. Let \( A = [v_1, v_2, \ldots, v_k] \), \( V = \text{Span}(v_1, v_2, \ldots, v_k) = R(A) = N(A^T) \). By solving the linear system \( A^T x = 0 \), we can generate an orthonormal set \( \{b_1, \ldots, b_{n-k+1}\} \). Then \( B = \{v_1, v_2, \ldots, v_k, b_1, \ldots, b_{n-k+1}\} \) is an orthonormal set of the whole vector space \( \mathbb{R}^n \). Since \( V \subset U \subseteq \mathbb{R}^n \), we can extract \( \{v_{k+1}, \ldots, v_m\} \) from \( \{b_1, \ldots, b_{n-k+1}\} \) such that \( \{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_m\} \) is an orthonormal set of \( U \).
2. It suffices to show \( V \cap W = \{0\} \). But this is obvious as \( \{v_1, v_2, \ldots, v_k\} \) and \( \{v_k, v_{k+1}, \ldots, v_m\} \) are orthogonal to each other, \( V \cap W \) must be \( \{0\} \).

\( \square \)
Problem 1. Find the eigenvalues of the following matrices:

\[
\begin{pmatrix}
3 & 2 \\
4 & 1
\end{pmatrix}
\quad
\begin{pmatrix}
6 & -4 \\
3 & -1
\end{pmatrix}
\quad
\begin{pmatrix}
2 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 3 & 0 \\
0 & 0 & 0 & 4
\end{pmatrix}
\]

Proof.

(a) \(\lambda_1 = 5, \lambda_2 = -1, \ x_1 = (1,1)^T, \ x_2 = (1, -2)^T\)

(b) \(\lambda_1 = 2, \lambda_2 = 3, \ x_1 = (1,1)^T, \ x_2 = (4, 3)^T\).

(k) \(\lambda_1 = \lambda_2 = 2, \lambda_3 = 3, \lambda_4 = 4, \ x_i = e_i, i = 1, 2, 3, 4\).

Problem 2. Show that the eigenvalues of a triangular matrix are the diagonal elements of the matrix.

Proof. Consider when \(\det(A - \lambda I) = 0\) for triangular matrix \(A\).

Problem 3. Let \(A\) be an \(n \times n\) matrix. Prove that \(A\) is singular if and only if \(\lambda = 0\) is an eigenvalue of \(A\).

Proof. If \(A\) is singular, then \(Ax = 0\) has a nontrivial solution, which is an eigenvector of \(A\) corresponding to \(\lambda = 0\). Conversely, if \(\lambda = 0\), then its corresponding eigenvector is a nontrivial solution to \(Ax = 0\).

Problem 4. Let \(A\) be a nonsingular matrix and let \(\lambda\) be an eigenvalue of \(A\). Show that \(1/\lambda\) is an eigenvalue of \(A^{-1}\).

Proof. Since \(A\) is nonsingular, \(A^{-1}\) exists and \(A\) only has nonzero eigenvalues.

\[Ax = \lambda x \iff A^{-1}Ax = \lambda A^{-1}x \iff A^{-1}x = \lambda^{-1}x\]