Section 6.1

8. We have $Av = \lambda v$ for some nonzero $v$. Then

$$A(Av) = A(\lambda v) \Rightarrow A^2 v = \lambda A v = \lambda^2 v.$$  

Using $A^2 = A$, we have $Av = \lambda^2 v$. Hence $\lambda v = \lambda^2 v$ and $\lambda = \lambda^2$, $\lambda = 0$ or $1$.

9. We have $Av = \lambda v$ for some nonzero $v$. Then

$$A^{k-1}(Av) = A^{k-1}(\lambda v) \Rightarrow A^k v = \lambda A^{k-1} v = \lambda^2 A^{k-2} v = \cdots = \lambda^k v.$$  

Using $A^k = O$, we have $0 = \lambda^k v$. Hence $\lambda^k = 0$, $\lambda = 0$.

12. Eigenvalues of $A$ are roots of $\det(A - \lambda I) = 0$. Eigenvalues of $A^T$ are roots of $\det(A^T - \lambda I) = 0$. Since transposition does not change the determinant, 

$$\det(A - \lambda I) = \det((A - \lambda I)^T) = \det(A^T - \lambda I).$$  

Therefore, $\det(A - \lambda I) = 0$ and $\det(A^T - \lambda I) = 0$ have the same roots. Equivalently, $A$ and $A^T$ have the same eigenvalues.

14. Let $\lambda_1, \lambda_2$ be the two eigenvalues of $A$. 

$$\lambda_1 + \lambda_2 = \text{tr}A = 8, \quad \lambda_1 \lambda_2 = \det A = 12.$$  

Solving the equations gives $\{\lambda_1, \lambda_2\} = \{2, 6\}$.

31. Columns of $A$ all add up to a constant $\delta$ is equivalent to $eA = \delta e$, where 

$$e = (1, 1, \ldots, 1).$$  

Hence $A^T e^T = \delta e^T$, $\delta$ is an eigenvalue of $A^T$. Using Question 14, it is also an eigenvalue of $A$.

32. We have $Ax = \lambda_1 x$ and $A^T y = \lambda_2 y$. Then 

$$\lambda_2 x^T y = x^T A^T y = (Ax)^T y = \lambda_1 x^T y.$$  

Since $\lambda_1 \neq \lambda_2$, we have $x^T y = 0$.

Section 6.2

1(a). Let $Y = (y_1, y_2)^T$. The equation is 

$$Y' = AY, \quad A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}.$$  

The eigenvalues of $A$ are $\lambda_1 = 2, \lambda_2 = 3$. Corresponding eigenvectors are 

$$v_1 = (1, 1)^T, \quad v_2 = (1, 2)^T.$$
So the general solution is
\[
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = C_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.
\]

2(a). Let \( Y = (y_1, y_2)^T \). The equation is
\[
Y' =AY, \quad A = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}.
\]
The eigenvalues of \( A \) are \( \lambda_1 = 1, \lambda_2 = -3 \). Corresponding eigenvectors are
\[
v_1 = (1,1)^T, \quad v_2 = (1,-1)^T.
\]
So the general solution is
\[
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = C_1 e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\]
Using \( Y(0) = (3,1)^T \), we obtain \( (3,1) = C_1 (1,1)^T + C_2 (1,-1)^T \). So the solution is
\[
y_1 = 2e^t + e^{-3t}, \quad y_2 = 2e^t - e^{-3t}.
\]

4. Assume tank \( A \) contains \( y_1(t) \) grams of salt and tank \( B \) contains \( y_2(t) \) grams of salt at time \( t \). We have \( y_1(0) = 40, y_2(0) = 20 \), and the equation
\[
\begin{align*}
y_1' &= 4 \frac{y_2}{100} - 16 \frac{y_1}{100}, \\
y_2' &= 16 \frac{y_1}{100} - 16 \frac{y_2}{100}.
\end{align*}
\]
Let \( Y = (y_1, y_2)^T \). The equation is
\[
Y' =AY, \quad A = \frac{1}{25} \begin{pmatrix} -4 & 1 \\ 4 & -4 \end{pmatrix}.
\]
The eigenvalues of \( A \) are \( \lambda_1 = -\frac{2}{5}, \lambda_2 = -\frac{6}{5} \). Corresponding eigenvectors are
\[
v_1 = (1,2)^T, \quad v_2 = (1,-2)^T.
\]
So the general solution is
\[
Y = C_1 e^{-\frac{2}{5}t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{-\frac{6}{5}t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}.
\]
Using \( Y(0) = (40,20)^T \), we obtain \( (40,20) = C_1 (1,2)^T + C_2 (1,-2)^T \). So the solution is
\[
y_1 = 25e^{-\frac{2}{5}t} + 15e^{-\frac{6}{5}t}, \quad y_2 = 50e^{-\frac{2}{5}t} - 30e^{-\frac{6}{5}t}.
\]
5(b). The equation is
\[\begin{align*}
y_1' &= y_3, \\
y_2' &= y_4, \\
y_3' &= 2y_1 + y_4, \\
y_4' &= 2y_2 + y_3.
\end{align*}\]

Let \( Y = (y_1, y_2, y_3, y_4)^T \), we have
\[Y' = AY, \quad A = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
2 & 0 & 0 & 1 \\
0 & 2 & 1 & 0
\end{pmatrix}.\]

The eigenvalues and eigenvectors are
\[\begin{align*}
\lambda_1 &= -2, \quad v_1 = (1, -1, -2, 2)^T, \\
\lambda_2 &= 2, \quad v_2 = (1, 1, 2, 2)^T, \\
\lambda_3 &= -1, \quad v_3 = (-1, -1, 1, 1)^T, \\
\lambda_4 &= 1, \quad v_4 = (-1, 1, -1, 1)^T.
\end{align*}\]

So
\[Y = C_1 e^{-2t}v_1 + C_2 e^{2t}v_2 + C_3 e^{-t}v_3 + C_4 e^t v_4.\]