1. Let \( \{x, y\} \) be a basis of a subspace \( S \) of \( \mathbb{R}^n \) and \( A = 2xx^T + 3yy^T \). In what follows we consider \( \mathbb{R}^n \) as an inner product space with its usual scalar product.

(a) (6 pt) Show that \( N(A) = S^\perp \).

Solution 1

We have that

\[
z \in N(A) \iff Az = 0 \iff 2xx^T z + 3yy^T z = 0 \iff (2x^T z)x + (3y^T z)y = 0
\]

Since \( x, y \) are linearly independent, this in turn is equivalent to

\[
2x^T z = 3y^T z = 0
\]

i.e. \( x^T z = y^T z = 0 \). This means that \( z \) is orthogonal to both \( x \) and \( y \) and thus \( z \in S^\perp \) as \( \{x, y\} \) is a basis for \( S \). Hence \( N(A) = S^\perp \).

Solution 2

Note that

\[
A^T = 2(xx^T)^T + 3(yy^T)^T = 2(x^T)^T x + 3(y^T)^T y = 2xx^T + 3yy^T = A
\]

Moreover, since for any \( z \) we have \( Az = (2x^T z)x + (3y^T z)y \in S \), it follows that \( R(A) \subseteq S \).

Let \( B \) be the matrix with rows \( 2x^T \) and \( 3y^T \). Since \( x, y \) are linearly independent, the rank of \( B \) is 2 and hence for any \( \alpha, \beta \) there exists \( z \in \mathbb{R}^n \) such that

\[
Bz = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \iff 2x^T z = \alpha, \ 3y^T z = \beta
\]

But then \( Az = \alpha x + \beta y \), which implies that \( R(A) = S \), since \( \alpha, \beta \) are arbitrary.

By the Fundamental Subspaces theorem we conclude that

\[
N(A) = R(A^T)^\perp = R(A)^\perp = S^\perp
\]

(b) (4 pt) Suppose that \( ||x|| = 1 \) and \( x \) is an eigenvector of \( A \) with eigenvalue 2. Show that \( x \) and \( y \) are orthogonal.

Solution

We have \( Ax = 2x \). But

\[
A = 2x^T x + (3y^T x)y = 2||x||^2 x + (3y^T x)y = 2x + (3y^T x)y
\]

Thus we must have \( (3y^T x)y = 0 \) \( \Rightarrow y^T x = 0 \) and hence \( x, y \) are orthogonal.
2. (10 pt) Let $P_3$ denote the vector space of polynomials with real coefficients of degree less than 3. Consider the function that assigns to any $p(x) = a_2x^2 + a_1x + a_0 \in P_3$ the number

$$\|p\| = |a_0| + 2|a_1| + 4|a_2| \in \mathbb{R}$$

Show that this function defines a norm on $P_3$.

**Solution** Since $|a_0|, |a_1|, |a_2| \geq 0$ we have

$$\|p\| = |a_0| + 2|a_1| + 4|a_2| \geq 0$$

with equality if and only if $|a_0| = |a_1| = |a_2| = 0$, i.e. $a_0 = a_1 = a_2 = 0$ and $p(x) = 0$.

Now, since for any scalar $\beta \in \mathbb{R}$ we have $\beta p(x) = \beta a_2x^2 + \beta a_1x + \beta a_0$,

$$\|\beta p\| = |\beta a_0| + 2|\beta a_1| + 4|\beta a_2| = |\beta| (|a_0| + 2|a_1| + 4|a_2|) = |\beta| \cdot \|p\|$$

Finally, for the triangle inequality for two polynomials $p(x) = a_2x^2 + a_1x + a_0$ and $q(x) = b_2x^2 + b_1x + b_0$, we have, using the inequality $|\alpha + \beta| \leq |\alpha| + |\beta|$ for $\alpha, \beta \in \mathbb{R}$,

$$\|p + q\| = |a_0 + b_0| + 2|a_1 + b_1| + 4|a_2 + b_2| \leq$$

$$\leq |a_0| + |b_0| + 2(|a_1| + |b_1|) + 4(|a_2| + |b_2|) =$$

$$= |a_0| + 2|a_1| + 4|a_2| + |b_0| + 2|b_1| + 4|b_2| =$$

$$= \|p\| + \|q\|$$

Thus $\|p\|$ defines a norm on $P_3$. 
3. For any pair of vectors \( \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \), \( \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \) in \( \mathbb{R}^2 \) define

\[ \langle \mathbf{x}, \mathbf{y} \rangle = \frac{1}{4} x_1 y_1 + \frac{1}{9} x_2 y_2 \]

(a) Show that \( \langle \mathbf{x}, \mathbf{y} \rangle \) defines an inner product on \( \mathbb{R}^2 \).

\textbf{Solution} We have

\[ \langle \mathbf{x}, \mathbf{x} \rangle = \frac{1}{4} x_1^2 + \frac{1}{9} x_2^2 \geq 0 \]

with equality if and only if \( x_1 = x_2 = 0 \iff \mathbf{x} = \mathbf{0} \).

For symmetry, \( \langle \mathbf{x}, \mathbf{y} \rangle = \frac{1}{4} x_1 y_1 + \frac{1}{9} x_2 y_2 = 4 y_1 x_1 + 9 y_2 x_2 = \langle \mathbf{y}, \mathbf{x} \rangle \).

Finally, for bilinearity we have

\[ \langle \alpha \mathbf{x} + \beta \mathbf{y}, \mathbf{z} \rangle = \frac{1}{4} (\alpha x_1 + \beta y_1) z_1 + \frac{1}{9} (\alpha x_2 + \beta y_2) z_2 = \]

\[ = \alpha \left( \frac{1}{4} x_1 z_1 + \frac{1}{9} x_2 z_2 \right) + \beta \left( \frac{1}{4} y_1 z_1 + \frac{1}{9} y_2 z_2 \right) = \]

\[ = \alpha \langle \mathbf{x}, \mathbf{z} \rangle + \beta \langle \mathbf{y}, \mathbf{z} \rangle \]

Therefore \( \langle \mathbf{x}, \mathbf{y} \rangle \) defines an inner product on \( \mathbb{R}^2 \).
Let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(b) (5 pt) Apply the Gram-Schmidt process to the basis $\{e_1, e_2\}$ to find an orthonormal basis for $\mathbb{R}^2$ with respect to the inner product given by $(\ast)$.

**Solution** Since $\|e_1\| = \frac{1}{2}$ we have

$$u_1 = \frac{1}{\|e_1\|}e_1 = 2e_1$$

Now, since $\langle e_2, u_1 \rangle = 0$, the projection of $e_2$ onto $u_1$ is

$$p_1 = \langle e_2, u_1 \rangle u_1 = 0$$

and hence, since $\|e_2\| = \frac{1}{3}$,

$$u_2 = \frac{1}{\|e_2 - p_1\|}(e_2 - p_1) = \frac{1}{\|e_2\|}e_2 = 3e_2$$

$\{u_1, u_2\}$ is an orthonormal basis for $\mathbb{R}^2$.

(c) (5 pt) Find the vector projection of the vector $v = 6e_2$ onto the vector $u = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ with respect to the inner product given by $(\ast)$.

**Solution** We have

$$\langle v, u \rangle = 2, \quad \langle u, u \rangle = 2$$

and hence the projection is

$$q = \frac{\langle v, u \rangle}{\langle u, u \rangle}u = u$$