Chapter 1 Test B

1. After performing Gaussian elimination to the augmented matrix, we arrive at the reduced row echelon form

\[
\begin{pmatrix}
1 & -1 & 0 & -7 & 4 \\
0 & 0 & 1 & 3 & -1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

so the solutions are

\[
x_1 = 4 + x_2 + 7x_4 \\
x_3 = -1 - 3x_4
\]

for any \(x_2, x_4 \in \mathbb{R}\).

2. (a) A linear equation in three unknowns corresponds to a plane in \(\mathbb{R}^3\).

(b) There can be no solutions at all or infinitely many solutions. The solution space of the system will be the intersection of the two planes corresponding to the two linear equations. These planes could be parallel, giving no solutions at all. They could also intersect in a line or be the same and in both of these cases we get infinitely many solutions.

(c) If the system is homogeneous, then both planes corresponding to the equations pass through the origin, so we have at least one solution \(x = 0\). The planes could intersect only at the origin, in a line through the origin or be the same, so in general we have either one solution or infinitely many.

3. (a) There are infinitely many solutions, as in order to have more than one solution the pivot columns are less than \(n\), otherwise there is a unique solution. But then there is at least one free variable, meaning we get an infinity of solutions.

(b) Since \(A(x_1 - x_2) = Ax_1 - Ax_2 = b - b = 0\) and \(x_1 - x_2 \neq 0\), \(x_1 - x_2 \in N(A)\), so the nullspace of \(A\) is nonzero (as a vector space) and hence \(A\) is singular.

6. We have

\[
A \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = 3a_1 + a_2 + 4a_3 = b
\]

so the system is consistent.

7. We have

\[
A \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} = a_1 - 3a_2 + 2a_3 = 0
\]

thus

\[
\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \in N(A)
\]

and \(A\) is singular.
9. Let
\[ A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} \]
Then
\[ C = AB = \begin{pmatrix} 2 & 0 \\ 3 & 2 \end{pmatrix} \]
so \( C \) is not necessarily symmetric.

**Chapter 2 Test B**

2. (a) \( \det(A) = x^3 - x \).
(b) \( A \) is singular if and only if \( \det(A) = x^3 - x = 0 \), which is the case when \( x = 0, \pm 1 \).

4. Since \( A \) is nonsingular, \( \det(A) \neq 0 \). Then
\[ \det(A^T A) = \det(A^T) \det(A) = \det(A) \det(A) = \det(A)^2 > 0 \]
so \( A^T A \) is nonsingular with positive determinant.

7. We have \( \det(A - \lambda I) = 0 \) if and only if \( A - \lambda I \) is singular if and only if \( x \in N(A - \lambda I) \) for some \( x \neq 0 \) if and only if \( (A - \lambda I)x = 0 \) for some \( x \neq 0 \). But
\[ (A - \lambda I)x = 0 \iff Ax - \lambda Ix = 0 \iff Ax = \lambda Ix = \lambda x \]

**Chapter 3 Test B**

2. (a) \( S \) is a subspace of \( \mathbb{R}^2 \).
(b) \( S \) is not a subspace of \( \mathbb{R}^2 \) as
\[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in S, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in S \]
however
\[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin S \]

3. (a) Using Gaussian elimination, we find that a basis for \( N(A) \) is
\[ \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \]
so the dimension of \( N(A) \) is 3.
(b) A basis for \( \text{Col}(A) \) is
\[ \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix} \right\} \]
thus it has dimension 2 and the rank of \( A \) is 2.

6. (a) The zero matrix is symmetric, thus \( 0 \in S \). If \( A, B \in S \) are symmetric, i.e. \( A^T = A \) and \( B^T = B \), then \( (A + B)^T = A^T + B^T = A + B \), so \( A + B \in S \) and for any \( \alpha \in \mathbb{R} \) we have \( (\alpha A)^T = \alpha A^T = \alpha A \), so \( \alpha A \in S \). We conclude that \( S \) is a subspace of \( \mathbb{R}^{2\times2} \).
(b) For \( A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in S \) we have 
\[
A = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\]
and it is easy to check that 
\[
\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}
\]
is a basis for \( S \).

8. (a) Since \( \dim \mathbb{R}^3 = 3 \), any set of four vectors must be linearly dependent, so \( x_1, x_2, x_3, x_4 \) are linearly dependent.

(b) Since \( \dim \mathbb{R}^3 = 3 \), two vectors cannot span \( \mathbb{R}^3 \).

(c) By putting them in a matrix as columns and performing Gaussian elimination, one can find that 
\[-2x_1 + 3x_2 - x_3 = 0\]
thus \( x_1, x_2, x_3 \) are linearly dependent, do not span \( \mathbb{R}^3 \) and are not a basis for \( \mathbb{R}^3 \) (all of these conditions are equivalent, since we have three vectors and \( \dim \mathbb{R}^3 = 3 \)).

(d) By putting them in a matrix as columns and performing Gaussian elimination, one can find that \( x_1, x_2, x_3 \) are linearly independent, thus span \( \mathbb{R}^3 \) and are a basis for \( \mathbb{R}^3 \) (all of these conditions are equivalent, since we have three vectors and \( \dim \mathbb{R}^3 = 3 \)).