1 Homework 2

Exercise 5.4: We use induction on \( n \). Suppose that \( n = 0 \). The sum 
\[
\sum_{i=0}^{0} x^i = x^0 = 1.
\]
On the other hand, \( \frac{1}{1-x} = 1 \), so the base case holds. Suppose now that the statement is true for a given \( n \), we will show that it holds for \( n + 1 \).
\[
\sum_{i=0}^{n+1} x^i = \sum_{i=0}^{n} x^i + x^{n+1} = \frac{1-x^{n+1}}{1-x} + x^{n+1} = \frac{1-x^{n+2}}{1-x}
\]

Exercise 6.1 (i): \( 0 \notin (0,1) \) because \( 0 < 0 \) is false. \( 0 \in [0,1] \) because \( 0 \leq 0 \leq 1 \) is true. \( 0 \in [0,1] \) because \( 0 < 0 \) is true. \( 0 \notin (0,1) \) because \( 0 < 0 \) is false.

(ii) \( [a,b] \setminus (a,b) = \{a,b\} \).

(iii) If \( a \geq b \), then is clear that \( (a,b) = \emptyset \), for otherwise, if there is \( x \in (a,b) \) we would have \( a < x < b \) and therefore \( a < b \). Suppose then that \( (a,b) = \emptyset \) and let’s show that \( a \geq b \). Since \( (a,b) = \emptyset \), the real number \( \frac{a+b}{2} \notin (a,b) \). This means that \( \frac{a+b}{2} \leq a \) or \( b \leq \frac{a+b}{2} \). Therefore \( a + b \leq 2a \) or \( 2b \leq a + b \). It follows that \( a \geq b \).

Similarly, \( (a,b) = \emptyset \) if and only if \( a \geq b \), \( [a,b] = \emptyset \) if and only if \( a \geq b \), and \( [a,b] = \emptyset \) if and only if \( a > b \).

(iv) Given that \( a \leq b \), both \( a \) and \( b \) belong to \( [a,b] \). Since \( [a,b] \subseteq (c,d) \) we have that \( a \in (c,d) \) and \( b \in (c,d) \). This happens if and only if \( c < a \leq b < d \).

Exercise 6.4: Let \( P(x) = "x \in A", Q(x) = "x \in B", R(x) = "x \in C" \) and \( T(x) = (P(x) \land Q(x)) \lor (P(x) \land R(x)) \)

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Exercise 6.5: (i) Suppose that \( A \subseteq B \). \( B \subseteq A \cup B \) always holds, so we only need to show that \( A \cup B \subseteq B \). Let \( x \in A \cup B \). This means that \( x \in A \) or \( x \in B \). If \( x \in B \), we are done. So suppose that \( x \in A \). However, since \( A \subseteq B \) then \( x \in B \) again. For the other implication, suppose that \( A \cup B = B \). Then \( A \subseteq A \cup B = B \) shows that \( A \subseteq B \).

(ii) Same reasoning as (i).

Exercise 6.6: Suppose by contradiction that \( x \in A \) but \( x \notin C \). If \( x \in B \) then \( x \in A \cap B \subseteq C \), contradicting that \( x \notin C \).
Exercise 7.2: (i) False, pick $m = 2, n = 1$.
(ii) True, pick $m = 1, n = 1$.
(iii) True, pick $n = m$.
(iv) True, pick $m = 1$.
(v) True, pick $m = n$.
(vi) False, pick $m = n + 1$.

Problem 13: Induction on $n$. Case $n = 4$ follows from $4! = 24 > 16 = 2^4$. Suppose now that for a given $n \geq 4$, we know that $n! > 2^n$. Then we have the following chain of inequalities

$$(n + 1)! = (n + 1) \cdot n! \geq 5n! \geq 2n! > 2 \cdot 2^n = 2^{n+1}$$