1 Homework 5

Exercise 12.5 \((1 + x)^{2n} = \sum_{i=0}^{2n} \binom{2n}{i} x^i\). On the other hand \((1 + x)^{n} = (\sum_{j=0}^{n} \binom{n}{j} x^j)^2\). Since these two polynomials are equal, they must be equal coefficient by coefficient. The coefficient of the term \(x^n\) in \((1 + x)^{2n}\) is \(\binom{2n}{n}\). The coefficient of the term \(x^n\) in \((1 + x)^{n} \) from the above expansion is \(\sum_{j} \binom{n}{j} \binom{n}{n-j} = \sum_{j} \binom{n}{j}^2\).

Exercise 12.6 We have to show that \(\frac{r(r+1)...(r+n)}{n!}\) is an integer. However, by multiplying above and below by \(r!\) we see that this number is equal to \(\frac{(r+n)!}{n!}\), which is an integer.

Exercise 14.2 Fix bijections \(f : A \to Z\) and \(g : B \to Z\). We can construct a third function \(h : A \cup B \to Z\) by the following rule: \(h(x) = 2f(x)\) if \(x \in A\) and \(h(x) = 2g(x) + 1\) if \(x \in B \setminus A\). Clearly \(h\) is injective. Since \(A \cup B\) is not finite, it must be denumerable.

Exercise 14.3 For every positive integer \(n\), fix a bijection \(f_n : Z \to A_n\). Then define a map \(g : Z \times Z^+ \to \cup_n A_n\) by setting \(g(i, n) = f_n(i)\). We claim that \(g\) is bijective. Let’s start with injectivity. Suppose that \(g(i, n) = g(j, m)\). Since \(g(i, n) \in A_n\), \(g(j, m) \in A_m\), and \(A_n \cap A_m = \emptyset\), we must have \(n = m\). Then \(f_n(i) = f_n(j)\) and therefore also \(i = j\) by the injectivity of \(f_n\). Let’s prove surjectivity. Let \(y \in \cup_n A_n\). This means that \(y \in A_n\) for some \(n\). Since \(f_n\) is surjective, there is \(i \in Z\) such that \(f_n(i) = y\), therefore \(g(i, n) = y\). In conclusion, \(g\) is bijective, and since \(Z \times Z^+\) is denumerable, we can conclude the thesis.

Problem III.18 Let’s call the given function \(f\). Suppose that \(f((A, B)) = f((C, D))\). Then \(A \cup B = C \cup D\). Then \(A \subseteq A \cup B = C \cup D\). Since \(A \subseteq X\) and \(X\) is disjoint to \(Y\), we have that \(A \subseteq (C \cup D) \setminus Y = C\). By symmetry \(C \subseteq A\), and therefore \(A = C\). Again by symmetry, \(B = D\). Therefore we can conclude that \(f\) is injective. Let’s prove surjectivity. Let \(E \subseteq X \cup Y\) with \(|E| = k\). Define \(A = E \cap X\) and \(B = E \cap Y\). First notice that \(A \cup B = (E \cap X) \cup (E \cap Y) = E \cap (X \cup Y) = E\). Also, \(A \cap B = \emptyset\). Therefore \(|A| + |B| = |E| = k\). This proves that \((A, B)\) is in the domain of \(f\) and that \(f(A, B) = E\). Now, to conclude the combinatorial equality, simply equate the cardinality of the domain and the cardinality of the codomain.

\[
|\cup_i \mathcal{P}_i(X) \times \mathcal{P}_{k-i}(Y)| = \sum_i |\mathcal{P}_i(X)| \cdot |\mathcal{P}_{k-i}(Y)| = \sum_i \binom{|X|}{i} \cdot \binom{|Y|}{k-i}
\]

On the other hand

\[
|\mathcal{P}_k(X \cup Y)| = \binom{|X| + |Y|}{k}
\]
so we are done.

**Problem III.20** Let \( I \) be our set of ten integers, and consider the map \( f : \bigcup_{k=1}^{9} P_k(I) \to \mathbb{N}_{1015} \) given by \( f(A) = \sum_{x \in A} x \). This is well defined, since \( 97 + 98 + \cdots + 106 = 1015 \). The cardinality of the domain is \( |P(I) - 2| \), since we are removing \( I \) and \( \emptyset \). Therefore the cardinality is \( 2^{10} - 2 = 1022 \). By pigeonhole, there exists \( A \neq B \) such that \( f(A) = f(B) \). To finish the exercise we note that then also \( f(A \setminus B) = f(B \setminus A) \), because the sum on the intersection is the same. Since these two sets are disjoint, we are done.

**Problem A** (a) Suppose that \( n < m \). Then notice that by definition \( B_n \subseteq A_n \) and \( B_m \cap A_n = \emptyset \). Therefore \( B_n \cap B_m = \emptyset \).

(b) Here we notice that \( \cup_n A_n = \cup_n B_n \) and that the \( B_n \) are all disjoint. We are almost in the hypothesis of Exercise 14.3, except for the fact that it’s not clear that the \( B_n \) are all denumerable. However, one can modify the proof of Exercise 14.3 to hold under the condition that the \( A_n \) are countable and at least one is denumerable. This condition is satisfied in our case since \( B_1 = A_1 \) is denumerable.

(c) Consider the sets \( S_n = \{ A \subseteq \mathbb{Z}^+ : |A| \leq n \} \). Then our set is the union of all the \( S_n \). Each \( S_n \) is denumerable because we have a surjective function \( f : (\mathbb{Z}^+)^n \to S_n \) defined by \( f((i_1, \cdots, i_n)) = \{i_1, \cdots, i_n\} \).