

## Homework 9

**Exercise 21.1** The definition of addition is given by 
\[ [a] + [b] = [a + b]. \] To see that it’s well defined, pick \( a' \sim a \) and \( b' \sim b \). This means that \( m|a - a' \) and \( m|b - b' \). But then 
\[ m|a - a' + b - b' = a + b - a' - b' \] and therefore \( a + b \sim a' + b' \). The check for the subtraction is exactly analogous.

**Exercise 22.1** (i) reflexive, symmetric, transitive. The equivalence classes are the sets of even and odd integers.
(ii) symmetric
(iii) symmetric
(iv) reflexive, symmetric, transitive. The equivalence classes are \( \{0\} \) and \( \{n, -n\} \) for \( n \) positive integer.
(v) symmetric, transitive
(vi) symmetric
(vii) reflexive, symmetric, transitive. The equivalence classes are the lines \( x = k \) for some constant \( k \in \mathbb{R} \).
(viii) reflexive, symmetric, transitive. The equivalence classes are the circles centered at the origin \((0, 0) \in \mathbb{R}^2\).

**Problem V.3** By definition of base ten, \( n = \sum_{0 \leq i \leq k} a_k 10^i \). Now, \( 9|n \) if and only if \( n \equiv 0(9) \), and this happens if and only if \( \sum_{0 \leq i \leq k} a_k 10^i \equiv 0(9) \). Since however \( 10 \equiv 1(9) \) we have that \( n \equiv \sum_{0 \leq i \leq k} a_k (9) \) so that \( 9|n \) if and only if \( 9|\sum_{0 \leq i \leq k} a_k \).

**Problem V.17** (i) symmetric, transitive.
(ii) reflexive, symmetric
(iii) reflexive, symmetric, transitive. Only one equivalence class, the whole set \( \mathbb{Z}^+ \).
(iv) reflexive, symmetric, transitive. The equivalence classes are \( \mathbb{Z}^+ \) and \( \mathbb{Z}^- \).
(v) symmetric, transitive.
(vi) symmetric.

**Problem V.18** We will explicitly check one case to show how it works, and then just state the result. (i) is for instance well defined: pick \( a', b' \) such that \( a/b = a'/b' \). By definition, this means that \( ab' = a'b \). Now, \( f(a/b) = a^2/b^2 \) and \( f(a'/b') = (a')^2/(b')^2 \), so that we have to check that \( a^2/b^2 = (a')^2/(b')^2 \). Again by definition, this happens if and only if \( a^2(b')^2 = (a')^2b'^2 \), which is true since we know that \( ab' = a'b \).
(ii) not well defined
(iii) not well defined
(iv) not well defined
(v) well defined.

**Problem V.19** (i) is not well defined since \([0]_6 = [6]_6\) but \([1]_4 \neq [7]_4\).
(ii) is well defined. In fact, pick $a' \sim a$, meaning that $6|a - a'$. Then $4|2 \cdot (a - a')$ because $2|a - a'$, therefore $f$ is well defined.

(iii) is well defined. Again suppose that $6|a - a'$. We have to show that $4|a^2 - (a')^2 = (a - a')(a + a')$. Since $2|a - a'$ we get that $2|a + a'$, therefore $4$ divides the product, as desired.